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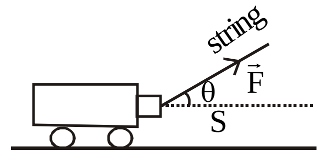
WORK, POWER, & ENERGY

Work

Let a force be applied on a body so that the body gets displaced. Then work is said to be done. So

'work is said to be done if the point of application of force gets displaced.'

work is also defined as *'the product of displacement and the force in the direction of displacement.'* Work is a scalar quantity.



Consider child pulling a toy car and walking. The direction of force is along the string. The car moves along the horizontal surface. Let θ be the angle between the direction of force and the horizontal surface. The displacement is caused by the horizontal component of force F (i.e. forces in the direction of displacement) and not by the entire force F. The horizontal component of force F is $F\cos\theta$. Work is defined as the product of displacement and the force in the direction of displacement. If S is the displacement, work done by the force,

$W = S \times F\cos\theta$.

$W = F S \cos\theta$. where θ is the angle between the direction of force and direction of displacement.

i.e, work done, $W = \vec{F} \bullet \vec{S}$

If θ is zero, W = F S, the maximum work done by the force. i.e. if the child pull the string along horizontal direction maximum effect is produced or maximum work will be done.

If θ = 90°, W = F S cos 90 = 0. No work is done. *If the force and displacement are mutually perpendicular, no work is done by that force.*

During circular motion, the velocity and hence displacement are tangential to the circular path. The centripetal force is towards the centre or the direction of it is normal to direction of displacement. Hence *work done by centripetal force during circular motion is zero.*

When θ = 180, i.e. force and displacement are in opposite direction, work done,

W = F S cos 180 = - F S. ie work done is negative.

Eg: A stone is moving vertically upwards. The gravitational force acts vertically downwards. Here the work done by the gravitational force is negative.

Unit of work in SI system

Work done, W = F S cosθ. When F - 1 N, S = 1 m and θ = 0, then W = 1 N m. This is called 1 joule.

The work done is said to be one joule if a force of one newton can displace a body through one metre in the direction of force.

The dimensional formula is ML^2T^{-2}.

Work done by a constant force

Consider a constant force F acting on a system so that the system gets displaced through a distance Δx,then work done by this constant force, $W = F\,\Delta x$.

Work - Energy Theorem.

According to work energy theorem, the change in kinetic energy of a particle is equal to the work done on it by the net force.

Consider a force F acting on a body of mass m so that its velocity changes from u to v in travelling a distance S.

Then work done, W = F S.

Now change in KE = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}m(v^2 - u^2)$

But $v^2 - u^2 = 2aS$

Therefore, change in KE = $= \frac{1}{2}m\,2aS = maS = FS =$ Work done

Thus, workdone = change in kinetic energy.

Energy

Energy is the capacity to do work. Energy is also a scalar quantity. Unit joule (J).
There are two types of energy - Kinetic Energy and Potential Energy.

Kinetic energy

The energy possessed by a body by virtue of its motion is called kinetic energy.

If a body of mass m moves with a velocity v, then kinetic energy, $K = \frac{1}{2}mv^2$.

Note: We know, kinetic energy, $K = \frac{1}{2}mv^2$. Multiply and divide RHS by mass m.

Then $K = \frac{m^2v^2}{2m} = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$; where p = mv, the momentum of the body. Also, $p = \sqrt{2mK}$

Work done by a variable force.

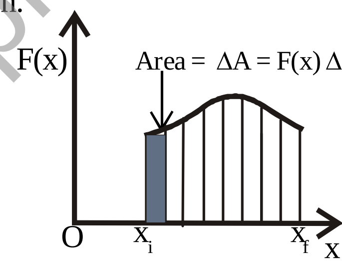
Consider a continuously varying force F(x) acting on a body and produces a very small displacement Δx each time. Since displacement is small, the force F(x) can be considered as constant. Then work done, ΔW = F(x) Δx.

Now a graph drawn connecting F(x) and displacement of body is as shown.

Now area of shaded portion, ΔA = F(x) Δx = Workdone ΔW

So the total workdone in displacing the body from x_i to x_f,

$$W = \int_{x_i}^{x_f} F(x)\Delta x$$ = the total area under the graph.



Work - Energy theorem for a variable force.

We know, kinetic energy, $K = \frac{1}{2}mv^2$

Change in KE wrt time, $\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\,2v\frac{dv}{dt} = m\frac{dv}{dt}v = mav = Fv = F\frac{dx}{dt}$

$\therefore dK = F\,dx.$

Integrating, $\int_{K_i}^{K_f} dK = \int_{x_i}^{x_f} F\,dx$ i.e. $K_f - K_i = \int_{x_i}^{x_f} F\,dx = W$

Thus change in kinetic energy is equal to workdone. This proves work energy theorem for a variable force.

Potential Energy

Potential energy is the energy possessed by a body by virtue of its position or state of strain.

For eg, a body at a height h above from the ground possess potential energy wrt earth due to its position. A compressed spring possess potential energy due to its state of strain.

A stretched bow, water stored in a dam, an elongated spring, a +ve charged particle kept near another +ve charge, a mango on a mango tree etc possess potential energy.

Gravitational potential energy of an object of mass 'm' kept at a height 'h' from the ground level.

The work done to raise the mass m to a height h against gravitational force

= displacement x force in the direction of displacement.

= h x mg = m g h.

This much of work done will be stored in it as potential energy.

Mechanical Energy: - The sum total of potential and kinetic energy of an object is called mechanical energy.

Problem 1. A car comes to a skidding stop in 20 m. During this process, the force on the cart due to the road is 400 N and is directly opposed to the motion. (a) How much work does the road do on the car? How much work does the car do on the road? {- 8000J, 0)

Problem 2. The momentum of a body is increased by 50%. By what percent does its KE change? {125%}

Problem 3. Two masses of 1 g and 4 g are moving with equal KE. What is the ratio of magnitudes of linear momenta? {1/2}

Problem 4. A body of mass 2 kg initially at rest moves under the action of an applied horizontal force of 7 N on a table with coefficient of kinetic friction = 0.1. Compute the (a) work done by the applied force in 10 s, (b) work done by friction in 10 s, (c) work done by the net force on the body in 10 s, (d) change in KE of the body in 10 s. {882 J, -246. 96 J, 635.04J, 635.04 J}

Problem 5. A body of mass 0.5 kg travels in a straight line with velocity $v = a x^{\frac{3}{2}}$ where a = 5 ms^{-2}. What is the work done by the net force during its displacement from x = 0 to x = 2 m? {50 J}

Conservative Force

A force is said to be conservative if the work done by the force is independent of the path but depends only on the initial and final positions. The work done by the conservative force in a closed path is zero. The force will be conservative, if it can be derived from a scalar quantity. Gravitational force and elastic spring force are conservative forces.

Law of conservation of energy.

According to law of conservation of energy, energy can neither be created nor destroyed. It can be changed from one form to another.

Proof of law of conservation of energy in case of a freely falling body.

Consider an object of mass ‘m’ kept at a height ‘h’ at position A.

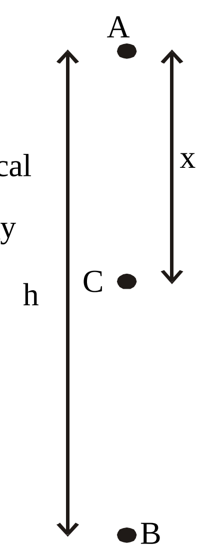
Case 1. At A.

Here energy is entirely potential and is equal to mgh.

$\therefore$ Mechanical energy at A = PE + KE = mgh + 0 = m g h ....(1)

Let the object be released. Just before touching the ground, its energy is entirely kinetic because h = 0 and PE = mgh = 0.

Case 2. At B.



Mechanical energy at B = PE + KE = 0 + $\frac{1}{2}mv^2$; where v is the final velocity or velocity at B.

Because initial velocity u = 0, a = g and S = h.

From equation of motion, $v^2 = u^2 + 2aS$ here $v^2 = 2gh$.

$\therefore \frac{1}{2}mv^2 = \frac{1}{2}m \times 2gh = mgh$ ....(2) which is equal to mechanical energy at B. So the potential energy at position A is completely converted to kinetic energy at position B.

Case 3. At C.

Now, consider an intermediate position C at a distance ‘x’ from A during the fall.
At C, its potential energy = mg(h - x) = m g h - m g x.

Kinetic energy = $\frac{1}{2}mv_C^2$

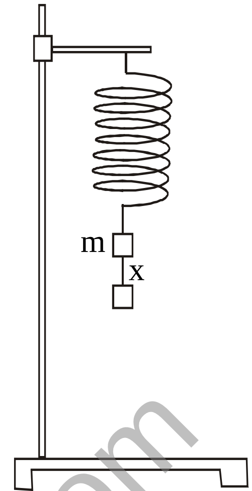
But $v_C^2 = u^2 + 2aS = 0 + 2gx$

$\therefore \frac{1}{2}mv_C^2 = \frac{1}{2}m \times 2gx = mgx$

$\therefore$ Mechanical energy at C = PE + KE = mgh - mgx + mgx = m g h ....(3)

From (1) , (2) and (3) we can see that the mechanical energy of a freely falling body remains constant.

Potential Energy of a spring.



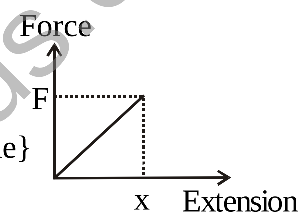
Consider a body of mass m attached at the end of a spring suspended from a rigid support. Now let the mass be pulled through a small distance x . Then the spring will try to come back to the initial position by giving an opposite force F. This force is called restoring force.

Now restoring force $F \propto x$ ie F = - kx. Here k is called **force constant or spring constant** of the spring. The -ve sign shows that the force is opposite to the displacement. Now let the spring be further pulled through a distance dx. Then work done, dw = F dx = kx dx.

Therefore the total work done in pulling the spring through a distance x,

$$W = \int dw = \int k\,x\,dx = \frac{1}{2} k\,x^2$$. This work done is stored as Potential Energy in the spring. Therefore the potential energy of a spring, $U = \frac{1}{2} k\,x^2$.

Graphical Method



Since $F \propto x$, a graph drawn connecting F and x is a straight line as shown. The area under the graph gives work done.

The area is given by $\frac{1}{2} x \times F$ {Since it is a triangle and area = 1/2 base x altitude}

But $F = k\,x$. $\therefore$ Work done, $W = \frac{1}{2} x \times kx = \frac{1}{2} kx^2$. This much work done will be stored in the spring as potential energy.

Mass energy equivalence

Albert Einstein showed that mass and energy are equivalent and they are connected by the relation $E = mC^2$.; where E is the energy when m gram of matter is converted and C is the velocity of light.

Power

Power is defined as work done per unit time or energy dissipated per unit time.

If an agency performs a work W in a time t, then $P = W/t$, average power, $P_{av} = \frac{dW}{dt}$; where dW is the small amount of work done within a very small interval or infinitesimal interval of time dt about that instant.

$$P_{inst} = \frac{dW}{dt} = \frac{d}{dt}(\vec{F} \bullet d\vec{x}) = \vec{F} \bullet \frac{d\vec{x}}{dt} = \vec{F} \bullet \vec{v}$$; where $\vec{v}$ is the instantaneous velocity, when the force is $\vec{F}$.

Power is a scalar quantity. Its dimensional formula is ML^2T^{-3}. In SI system, its unit is watt. $1 \text{ watt} = \frac{1 \text{ joule}}{1 \text{ second}}$

There is another unit of power called horse power (hp) 1 hp = 746 watt.

Kilowatt hour. It is the unit of electrical energy. 1 kilo watt hour = 1 x 1000 x 1J/1s x 3600 s = 3.6×10^6 joule.

Collissions.

A collision is said to have taken place if two moving objects strike each other or come close to each other such that the motion of one of them or both of them changes suddenly.

There are two types of collisions. (1) Elastic collision (2) Inelastic collision

Elastic collision

Elastic collision is one in which both momentum and kinetic energy is conserved.

Eg: (1) collision between molecules and atoms (2) collision between subatomic particles.

Characteristics of elastic collision

(1) Momentum is conserved (2) Total energy is conserved (3) K. E. is conserved (4) Forces involved during collision are conservative forces (5) Mechanical energy is not conserved if it converted into any other form like sound, light heat etc.

Inelastic collision

Inelastic collision is one in which the momentum is conserved, but KE is not conserved.
Example. (1) Mud thrown on a wall (2) Any collision between macroscopic bodies in energy day life.

Characteristics of inelastic collision

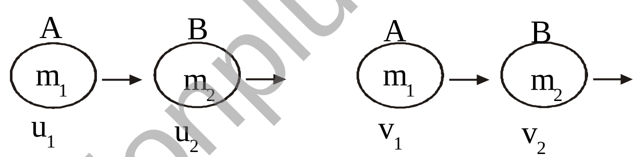
(1) Momentum is conserved (2) Total energy is conserved (3) K.E. is not conserved (4) Forces involved are not conservative (5) Part or whole of the KE is converted into other forms of energy like heat, sound, light etc.

Collisions in one dimension

If the two objects moves along the same line before and after collision, that is considered as collision in one dimension.

Let an object of mass m_1 moving with a velocity u_1, collide with a body of mass m_2 moving with velocity u_2.

Now by conservation of linear momentum.



Before collision After collision

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad .... (1)$$

$$m_1u_1 + m_1v_1 = m_2v_2 + m_2u_2$$

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad .... (2).$$

This is an elastic collision, hence K.E. is conserved.

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \quad .... (3)$$

$$\frac{1}{2}m_1u_1^2 - \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_2u_2^2.$$

$$\frac{1}{2}m_1(u_1^2 - v_1^2) = \frac{1}{2}m_2(v_2^2 - u_2^2)$$

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad .... (4)$$

$$m_1(u_1 + v_1)(u_1 - v_1) = m_2(v_2 + u_2)(v_2 - u_2) \quad ..... (5)$$

(5)/ (2) gives,

$$\frac{m_1(u_1 + v_1)(u_1 - v_1)}{m_1(u_1 - v_1)} = \frac{m_2(v_2 + u_2)(v_2 - u_2)}{m_2(v_2 - u_2)}$$

$$u_1 + v_1 = v_2 + u_2 \quad .... (6) \qquad v_1 - v_2 = -(u_1 - u_2) \qquad v_2 = v_1 + u_1 - u_2$$

To find v_1 and v_2

From (6), $v_2 = v_1 + u_1 - u_2$

substitute in (1). $m_1u_1 + m_2u_2 = m_1v_1 + m_2(v_1 + u_1 - u_2)$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_1 + m_2u_1 - m_2u_2$$

$$m_1u_1 + m_2u_2 - m_2u_1 + m_2u_2 = m_1v_1 + m_2v_1$$

$$2m_2u_2 + (m_1 - m_2)u_1 = v_1(m_1 + m_2).$$

$$v_1 = \frac{(m_1 - m_2)u_1 + 2m_2u_2}{m_1 + m_2} \quad .... (7)$$

Similarly,

$$v_2 = \frac{2m_1u_1 + u_2(m_2 - m_1)}{m_1 + m_2} \quad .... (8)$$

Discussion

Case - 1 $m_1 = m_2 = m$. Substitute those values in (7) and (8), we have

$$v_1 = \frac{2mu_2}{2m} = u_2$$

$$v_2 = \frac{2mu_1}{2m} = u_1.$$

i.e.. bodies exchange their velocities.

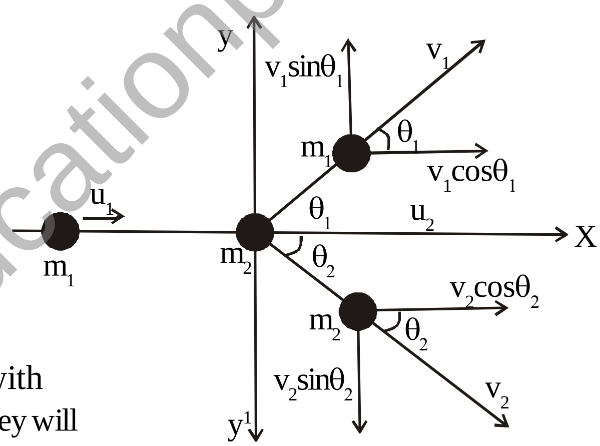
Case - 2 $m_2 >> m_1$ and $u_2 = 0$ ie; $m_1 - m_2 \approx -m_2$ $m_1 + m_2 \approx m_2$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 = \frac{-m_2 u_1}{m_2} = -u_1$$

$$v_2 = \frac{2 \times 0 \times u_1}{m_2} = 0.$$

The second body remains at rest while the first body rebounds with the same velocity.

Collisions in Two Dimensions



Consider two bodies of masses m_1 and m_2 moving with velocities u_1 and u_2 along parallel lines. If $u_1 > u_2$ they will collide. Let v_1 and v_2 be their velocities after collision along directions θ_1 and θ_2. v_1 and v_2 can be resolved in to $v_1\cos\theta_1$ and $v_2\cos\theta_2$ parallel to *x* axis and $v_1 \sin\theta_1$ $v_2 \sin\theta_2$ parallel to y axis.

By conservation of momentum parallel to X-axis. $m_1u_1 + m_2u_2 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2$

By conservation of momentum parallel to y-axis. $m_1v_1\sin\theta_1 + m_2v_2\sin\theta_2 = 0 + 0 = 0$

By conservation of kinetic energy $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$.

Problem 5. A block of mass 2 kg is dropped from a height of 40 cm on a spring whose force constant is 1960 N/m. What will be the maximum compression of the spring? {10 cm}

Problem 6. A trolley of mass 300 kg carrying a sandbag of 25 kg is moving uniformly with a speed of 27 km/h on a frictionless track. After a while, sand starts leaking out of a hole on the floor of the trolley at the rate of 0.05 kg/s. What is the speed of the trolley after the entire sand bag is empty?

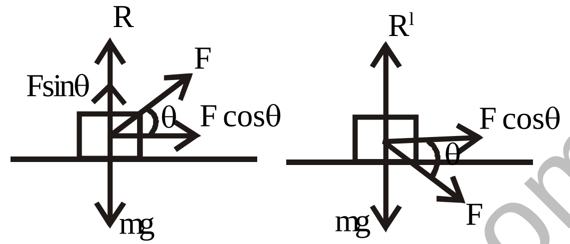
Problem 7. A trolley of mass 200 kg moves with a uniform speed of 36 km/h on a frictionless track. A child of mass 20 kg runs on the trolley from one end to the other (10 cm away) with a speed of 4 m/s relative to the trolley in a direction opposite to its motion, and jumps out of the trolley. What is the final speed of the trolley? How much has the trolley moved from the time the child begins to run? {10.36 m/s, 25.9 m}

Problem 8. A moving body of mass m_1 strikes a stationary body of mass m_2. The masses m_1 and m_2 should be in the ratio m_1/m_2 so as to decrease the velocity of the first body by 1/2 in a perfectly elastic impact. What

is the ratio m_1/m_2?
Problem 9. Why is it easier to pull a body than to push it?

Suppose a force F is applied to pull a block of mass m. The force can be resolved into two components. If R is the normal reaction, N + F sinθ = mg Therefore, N = mg - F sinθ............(1)

If a force F is applied to push the same block of mass m, Normal reaction, R^1 = mg + F sinθ...........(2) Since frictional force is directly proportional to normal reaction, from (1) and (2) we can see that friction in case of pull will be less than in case of push. Therefore it is easier to pull than to push.



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GRAVITATION

Newton's Universal law of gravitation.

"Every particle of matter in this universe attracts every other particle with a force which varies directly as the product of their masses and inversely as the square of the distance between them".

Consider two bodies of mass m_1 and m_2 separated by a distance r.
Then the force of attraction F between them,

$$F \propto m_1 m_2.$$

$$F \propto \frac{1}{r^2}$$

$$\therefore \quad F \propto \frac{m_1 m_2}{r^2}$$

Or $F = \dfrac{G\, m_1 m_2}{r^2}$; where G is a constant which has the same value everywhere and is known as Universal constant of gravitation or gravitational constant.

Definition of G .

If $m_1 = m_2 = 1$ kg and r = 1 m, then G = F newton.

Thus gravitational constant may be defined as the force of attraction between two bodies each of unit mass, separated by a distance of unity.

Unit of G .

$$G = \frac{Fr^2}{m_1 m_2} = N\, m^2/kg^2.$$

** Value of G = 6.67×10^{-11} N m^2/kg^2.
** Dimension of G.

$$G = \frac{Fr^2}{m_1 m_2} = M^{-1} L^3 T^{-2}.$$

Acceleration due to gravity. OR Relation between g and G.

Force between earth and a body near it is called gravity.

The uniform acceleration produced in a freely falling body due to gravitational pull of the earth is known as acceleration due to gravity.

Consider a body of mass m, placed on the surface of earth of mass M and radius R.

Then force of gravity acting on the body, F = m g .
Also, according to Newton's gravitational law,

$$F = \frac{GMm}{R^2}$$

From (1) and (2) ; $mg = \frac{GMm}{R^2}$ Or $g = \frac{GM}{R^2}$

This is the relation between g and G.

Note: g is independent of the mass of the body. *So different objects on the surface of earth, irrespective of their masses, experience the same acceleration towards the centre of the earth.*

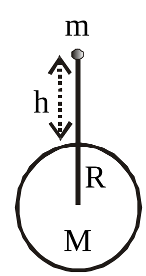
(i) **Variation of 'g' with altitude (height)**

Consider a body of mass **m** lying on the surface of earth of mass M and radius R. Let g be the value of acceleration due to gravity on the surface of earth.

$$\therefore \quad g = \frac{GM}{R^2} \quad \text{.........................(1)}$$

Now let the body be taken to a height 'h' above the surface of earth where the value of acceleration due to gravity is g_h.

$$\text{Then } g_h = \frac{GM}{(R+h)^2} \quad \text{.........................(2)}$$



$$\frac{(2)}{(1)} \Rightarrow \frac{g_h}{g} = \frac{GM}{(R+h)^2} \times \frac{R^2}{GM}$$

$$\therefore \quad \frac{g_h}{g} = \frac{R^2}{(R+h)^2}$$

$$\text{i.e. } \frac{g_h}{g} = \frac{R^2}{R^2\left(1+\frac{h}{R}\right)^2} = \frac{1}{\left(1+\frac{h}{R}\right)^2} = \left(1+\frac{h}{R}\right)^{-2}$$

Here if h <<<<<< R, then squares and higher powers of h/R can be neglected.

$$\therefore \quad \frac{g_h}{g} = 1 - \frac{2h}{R} \qquad \text{So} \quad g_h = g\left(1 - \frac{2h}{R}\right) = g - \frac{2gh}{R}$$

$$\text{Hence} \quad g_h - g = -\frac{2hg}{R} \quad \text{or} \quad g - g_h = \frac{2hg}{R}$$

Here $g - g_h$ gives the decrease in value of g. Since the value of g at a given place on earth is constant and R is also constant, $g - g_h \propto h$. So as h increases, g_h must decrease as g is a constant. *Thus the value of acceleration due to gravity decreases with increase in height above the surface of earth.*

Loss of weight at a height h (h <<<<< R)

$$\text{We know, } g_h = g\left(1 - \frac{2h}{R}\right)$$

$$\therefore \quad mg_h = mg\left(1 - \frac{2h}{R}\right) = mg - \frac{2mgh}{R}$$

$$\text{Now} \quad mg_h - mg = -\frac{2mgh}{R} \qquad \text{i. e. Loss of weight} = \frac{2mgh}{R}$$

ii) **Variation of g with depth**.

Assume the earth to be homogeneous sphere (having uniform density) of radius R and mass M. Let ρ be the mean density of earth. Consider a body of mass m lying on the surface of earth at a place where g is the acceleration due to gravity.

$$\text{Then } g = \frac{GM}{R^2} = \frac{G\frac{4}{3}\pi R^3\rho}{R^2}$$

$$\text{i.e. } g = \frac{4}{3}\pi R\rho G \quad ....(1)$$

Now let the body be taken to a depth **d** below the surface of earth where the value of acceleration due to gravity is g_d.

Here the force of gravity acting on the body is only due to the inner solid sphere of radius R - d.

So $g_d = \frac{GM^1}{(R-d)^2} = \frac{G}{(R-d)^2} \times \frac{4}{3}\pi(R-d)^3\rho$; where M^1 is the mass of the inner solid sphere of radius R - d.

$$\text{i.e. } g_d = \frac{4}{3}\pi(R-d)\rho G \quad ....(2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{g_d}{g} = \frac{\frac{4}{3}\pi(R-d)\rho G}{\frac{4}{3}\pi R\rho G} = \frac{R-d}{R}$$

$$\frac{g_d}{g} = 1 - \frac{d}{R} \qquad \text{Or} \qquad \underline{\underline{g_d = g\left(1 - \frac{d}{R}\right)}}$$

$$\text{Or} \quad g - g_d = \frac{gd}{R}$$

Here $g - g_d$ gives the decrease in value of g with depth. Since g is a constant at a given place of earth and R is also a constant, $g - g_d \propto d$. From the above equation it is clear that if d increases, g_d must decrease, because g is a constant. *Thus the value of acceleration due to gravity decreases with the increase of depth.*

Weight of a body at the centre of earth.

At a depth d below the free surface of earth, $g_d = g\left(1 - \frac{d}{R}\right)$

At the centre of earth, d = R $\quad \therefore \quad g_d = g\left(1 - \frac{R}{R}\right) = 0$

If m is the mass of a body lying at the centre of earth, then its weight = m g_d = 0.

Hence the weight of a body at the centre of the earth is zero.

Comparison of height and depth for the same change in 'g'.

As we have seen the value of g decreases as we go above the surface of earth or below the surface of earth. i.e. *the value of g is maximum on the surface of earth.*

$$g_h = g\left(1 - \frac{2h}{R}\right) \text{ and } g_d = g\left(1 - \frac{d}{R}\right) = 0 \qquad \text{i.e. } \underline{\underline{2h = d}}$$

Thus the value of g at a height h is same as the value of g at a depth d = 2h. But this is true only if h is very small.

Gravitational field of earth

Gravitational field of earth is the space around the earth where its gravitational influence is felt.

Intensity of gravitational field at a point:

It is defined as the force experienced by a body of unit mass placed at that point.

Gravitational field intensity of earth:

If a body of mass m is placed on the surface of earth of mass M and radius R, then the gravitational force experienced by the body, $F = \frac{GMm}{R^2}$

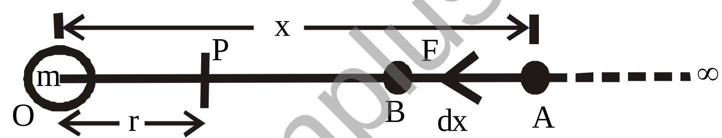
Intensity of gravitational field $= \frac{F}{m} = \frac{GM}{R^2} = g$

Thus intensity of gravitational field near the surface of earth is equal to the acceleration due to gravity at the place.

NB **Gravitational Potential**

Gravitational potential at any point inside a gravitational field is defined as the work done in taking a unit mass from infinity to that point.

Consider a body of mass m situated at O as in figure.



The body has a gravitational field surrounding it on all sides. The force F acting on a unit mass placed at a point A, at a distance x from O is given by

$$F = \frac{Gm \times 1}{x^2} = \frac{Gm}{x^2} \text{ along AO.}$$

Now let the unit mass be displaced through a small distance dx from A to B. If dw is the work done by the gravitational force in moving the unit mass from A to B,

$$dw = \vec{F} \bullet d\vec{x} = F\,dx\cos\theta = F\,dx \quad \{\text{since F and dx are in the same direction}\}$$

or $\quad dw = \frac{Gm\,dx}{x^2}$

Therefore the total work done in taking the unit mass from infinity to P, which is also the gravitational potential V at point P is given by

$$V = \int_0^W dw = \int_\infty^r \frac{Gm\,dx}{x^2} = Gm\int_\infty^r x^{-2}dx = Gm\left[\frac{x^{-2+1}}{-2+1}\right]_\infty^r$$

i.e, $\quad V = Gm\left[-\frac{1}{x}\right]_\infty^r = Gm\left[-\frac{1}{r} - \frac{1}{-\infty}\right]$

$$\therefore \; V = \frac{-Gm}{r}$$

The negative sign shows that the gravitational force is attractive in nature.

Note:(1) If a large mass M (may be earth) is doing work in taking a small mass m from infinity to a point P distant r from M, then gravitational potential energy, $V = \frac{-GMm}{r}$

(2) Gravitational PE is maximum and zero at infinity. At all other points, it is less than zero. It is a scalar quantity.

Geostationary or Synchronous satellite

A geostationary satellite is an artificial satellite orbiting the earth so that its time period is synchronous (same) with that of the earth. The orbit in which they stay is called **parking orbit or geostationary orbit**.

Uses: Communication, weather forcasting etc.

Orbital velocity of a satellite.

The velocity with which a satellite moves in its closed orbit is called orbital velocity.

Consider a satellite of mass m moving with a velocity v around the earth of mass M and radius R. Let r is the orbital radius of the satellite.

Here the centripetal force for rotation of satellite is given by gravitational force.

$$\text{i.e,} \quad \frac{mv^2}{r} = \frac{GMm}{r^2} \qquad \text{or} \quad v^2 = \frac{GM}{r}$$

$$\text{or} \quad v = \sqrt{\frac{GM}{r}} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

If the satellite is at a height h above the surface of earth, then, $r = R + h$

$$\therefore \quad v = \sqrt{\frac{GM}{(R+h)}} \quad \ldots\ldots\ldots\ldots(2)$$

$$\text{But } g = \frac{GM}{R^2} \qquad \text{Therefore,} \quad GM = gR^2.$$

Therefore, from (2), $v = \sqrt{\dfrac{gR^2}{(R+h)}}$

If the satellite is close to the earth, $(R+h) \approx R$

Therefore, orbital velocity, $v = \sqrt{\dfrac{GM}{R}} = \sqrt{gR}$

This orbit in which the satellite revolve is called **minimum orbit**. The velocity corresponding to minimum orbit is called **first cosmic velocity**.

The value of first cosmic velocity is 7.92 km/s.

Time period of a satellite.

It *is the time taken by the satellite to revolve once around the earth*.

If a satellite moves with a velocity v in an orbit of radius r, then time period, $T = \dfrac{2\pi r}{v}$. But $v = \sqrt{\dfrac{GM}{R}}$

$$\therefore \quad T = \frac{2\pi r}{\sqrt{\frac{GM}{R}}} = 2\pi\sqrt{\frac{r^3}{GM}} \quad \ldots\ldots\ldots\ldots(1)$$

If the satellite is at a height h from earth, $r = R + h$.

$$\therefore \quad T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} \quad \ldots\ldots\ldots\ldots(2)$$

For minimum orbit, $T = 2\pi\sqrt{\dfrac{R^3}{GM}} \quad \ldots\ldots\ldots\ldots(3)$

Also since $g = \frac{GM}{R^2}$; $GM = gR^2$.

$$\therefore T = 2\pi\sqrt{\frac{R^3}{gR^2}} = 2\pi\sqrt{\frac{R}{g}} \quad .....(4)$$

Escape Velocity.

Escape velocity is defined as the least velocity with which a body must be projected vertically upwards so that it may escape the gravitational pull of earth.

Consider a body of mass m placed on the surface of earth of mass M and radius R. Let the body be projected upwards with a velocity v_e so that it escapes from the gravitational field of earth.

Then KE of the body near the surface of earth = ½ m v_e^2.

PE of the body on the surface of earth $PE = -\frac{GMm}{R}$

Total energy of the body near the surface = $\frac{1}{2}mv_e^2 - \frac{GMm}{R}$

At infinity, PE = KE = 0.

$$\therefore \frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0.$$

$$\therefore \frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

$$\therefore v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2gR^2}{R}} = \sqrt{2gR} \quad \left[\because g = \frac{GM}{R^2}\right]$$

The escape velocity of earth is 11.2 km/s.

NB **Kepler's laws of planetary motion**

Law 1. Every planet revolves around the sun in an elliptical orbit with sun at one of its foci.

Law 2. The radius vector joining sun and the planet sweeps equal areas in equal intervals of time. That is areal velocity of the planet is constant.

Law 3. The square of the period of a planet is directly proportional to the cube of semi major axis of the orbit.

ie $T^2 \propto a^3$.

$$\therefore \left(\frac{T_1}{T_2}\right)^2 \propto \left(\frac{a_1}{a_2}\right)^3$$

Homework Problems.

1. Calculate the velocity of escape of an artificial satellite projected from the earth. Given, the mass of the earth = 5.9 x 10^{24} kg. Radius of the earth = 6370 km. Gravitational constant, G = 6.67 x 10^{-11} Nm^2/kg^2. [11.11 km/s]

2. An earth satellite makes a complete circle around the earth in 90 minutes. Assuming the orbit to be circular, calculate the height of the satellite above the earth. Radius of earth = 6370 km, g = 9.80 m/s^2. [276m]

3. Estimate the height above the earth at which the geostationary satellite is moving round the earth. Radius of earth = 6400 km. Mass of the earth = 6 x 10^{24} kg. [35912 km]

4. Determine the escape velocity of a body from the moon. Radius of the moon = 1.74 x 10^6 m. Mass of moon = 7.36 x 10^{22} kg. G = 6.67 x 10^{-11} Nm^2kg^{-2}. [2.38 km s^{-1}.]

5. Jupiter has a mass 318 times that of the earth, and its radius is 11.2 times the radius of the earth. Estimate the escape velocity of a body from the surface of Jupiter, given that the escape velocity form the earth's surface is 11.2 km/s. [59.7 km/s]

6. Assuming the earth to be a perfect sphere of uniform mass density, how much would a body weigh half way down the center of the earth if it weighed 250 N on the surface? [125 N]

7. An artificial satellite of mass 200 kg revolves round the earth in an orbit of average radius 6670 km. Calculate the orbital KE, gravitational potential energy and total energy of the satellite in the orbit. Given mass of earth = 6.4 x 10^{34} kg; Gravitational constant = 6.67 x 10^{-11} Nm^2/kg^2 [6 x 10^{19} J, –12 x 10^{19} J, – 6 x 10^{19} J]

8. World's first man –made artificial satellite Sputnik I was orbiting the earth at a distance of 900 km. Calculate its velocity. Radius of the earth = 6370 km. g = 9.8 m/s^2. [7.396 km/s]

9. If a satellite is to circle the earth at 1000km. above the surface, with only the attraction of the earth acting on it, at what speed must it travel? Mass of earth = 5.9 x 10^{24} kg. Radius of earth = 6370 km.
G = 6.67 x 10^{-11} Nm^2/kg^2. [7.308km/h]

10. The moon moves around the earth with a period of 27.3 days. Find the acceleration of the moon towards the center of the earth assuming that the orbit is circular with a radius of 384000 km. [0.002726 m/s]

Energy of a satellite in its orbit.

The kinetic energy of the satellite in a circular orbit at height h from surface of earth with speed v is

$$K.E = \frac{1}{2}mv^2 .$$

But $v = \sqrt{\frac{GM}{R+h}}$

$$\therefore kE = \frac{1}{2}m\frac{GM}{(R+h)} .$$

The potential energy at distance (R+h) from the centre of the earth is $p.E = \frac{-GMm}{(R+h)}$.

$$\therefore \text{Total energy} \quad E = k.E. + p.E \quad = \frac{1}{2}\frac{GMm}{(R+h)} + \frac{-GMm}{(R+h)} .$$

$$E = -\frac{1}{2}\frac{GMm}{(R+h)}$$

....

SYSTEM OF PARTICLES AND ROTATIONAL MOTION

Rigid Body

A body is said to be rigid if the distance between any two points in the body always remains the same even under the action of external forces.

Also a body is said to be rigid if it has a fixed geometrical shape and size which does not change during motion.

Rotational Motion

A body is said to be in rotational motion if all the particles of the body moves in circles, the centres of which lies in a straight line called axis of rotation.

Eg: Motion of the blades of a fan, motion of a top, motion of a door etc.

Distinction between translational motion and rotational motion of a rigid body.

In pure translational motion of a body, all the particles in it will have the same velocity at an instant of time. Eg:: a rectangular block sliding down an inclined plane.

Consider a body fixed or pivoted along a straight line. Then it is impossible for the body to make translational motion. The only possible motion of such a rigid body is rotation.

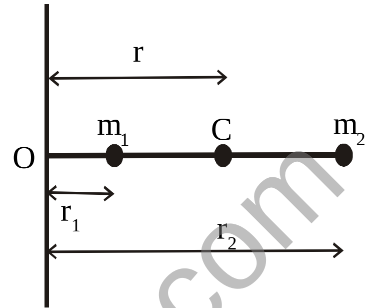
Thus we can conclude that the motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation. The motion of a rigid body which is pivoted or fixed in some way is rotation.

Centre of mass

Consider the motion of a system or body consisting of a large number of particles. There is one point in the body, which behaves as though the entire mass of the body were concentrated there. Also if external forces are applied on this point, the body moves as if the external forces are acting on the whole body. This point is called center of mass of the body.

Hence the center of mass of a system is the point where all the mass of the system may be assumed to be concentrated and where the resultant of all the external forces acts.

Centre of mass of a two particle system



Consider a system of two particles with masses m_1 and m_2 having their position vectors $\vec{r}_1$ and $\vec{r}_2$ from some arbitrary origin O. The center of mass will be at a point Q, whose radius vector is given by

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

If the particles have the same mass, $m_1 = m_2 = m$, then $\vec{r} = \frac{m\vec{r}_1 + m\vec{r}_2}{2m} = \frac{r_1 + r_2}{2}$.

Thus if the particles are having same mass, the centre of mass lies exactly midway between them.

Velocity of center of mass

The position vector of center of mass of a two particle system is given by $\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$; where $M = (m_1 + m_2)$.

$\therefore M\vec{r} = m_1\vec{r}_1 + m_2\vec{r}_2$

Differentiating w r t time, $\frac{d}{dt}(M\vec{r}) = \frac{d}{dt}(m_1 r_1 + m_2 r_2)$

$$M\frac{dr}{dt} = m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt}$$

$\therefore M v_{CM} = m_1 v_1 + m_2 v_2$

Therefore velocity of centre of mass, $v_{CM} = \frac{m_1 v_1 + m_2 v_2}{M}$

** Acceleration of center of mass.

We know,

The velocity of centre of mass, $v_{CM} = \frac{m_1 v_1 + m_2 v_2}{M}$; where v_1 and v_2 are the velocities of the particles.

Differentiating w r t time,

$$\frac{dv_{CM}}{dt} = \frac{1}{M}\frac{d}{dt}(m_1 v_1 + m_2 v_2)$$

$$= \frac{1}{M}\left[m_1\frac{dv_1}{dt} + m_2\frac{dv_2}{dt}\right]$$

ie $a_{CM} = \frac{m_1 a_1 + m_2 a_2}{M}$; where a_1 and a_2 are the accelerations of the particles.

If F_1 and F_2 are the external forces acting on the particles. Then $a_{CM} = \frac{F_1 + F_2}{M}$

Therefore, $M a_{CM} = F_1 + F_2$.

Here $M a_{CM}$ is the total force acting on the body or system. But total force is the sum of internal and external forces. Since internal force is zero, the total force acting on the system is external force only.

Therefore, the centre of mass moves as if it were a particle of mass equal to the total mass of the system and all the external forces are acting on it.

Centre of mass of an N particle system

Consider a system of N particles of masses m_1, m_2, m_3 .... having radius vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$,.........

Then the total mass of the system, $M = m_1 + m_2 + m_3 + \ldots\ldots\ldots$

Then, radius vector of centre of mass, $\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots}{M}$

Velocity of centre of mass, $v_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots\ldots}{M} = \frac{\sum mv}{M}$

Acceleration of centre of mass, $a_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots\ldots\ldots}{M}$

ie $M a_{CM} = m_1 a_1 + m_2 a_2 + \ldots\ldots\ldots$

$= F_1 + F_2 + F_3 + \ldots\ldots\ldots$

$\therefore\ M a_{CM} = F_{tot}$; where F_{tot} is the total external force acting on the system.

Examples of centre of mass motion.

(1) **Explosion of a shell in flight.**

Consider a shell projected upwards. This shell will follow a parabolic path. Now let the shell explode during flight. After explosion, the fragments travel in their own parabolic path. Since the force of explosion are all internal, the center of mass of the system will continue to follow the same parabolic path of the shell before explosion.

(2) **Motion of earth – moon system**

Moon moves round the earth in circular orbit, and earth moves round the sun in an elliptical orbit. Or we can say that the center of mass of the earth – moon system moves in an elliptical orbit round the sun. Hence the force of attraction between earth and moon is internal to earth – moon system while sun's attraction on both earth and moon are external.

Linear momentum of a system of particles.

For a system of n particles, the total linear momentum of the system is equal to the vector sum of momentum of all individual particles.

i.e, $p = p_1 + p_2 + p_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + p_n$.

Now from Newton's second law, $F = \frac{dp}{dt}$; where F is the total external force.

If $F = 0$, $\frac{dp}{dt} = 0$. Therefore, p will be a constant i.e, linear momentum will be conserved.

Examples:-

1) **Decay of nucleus**: Consider the decay of a parent nuclei at rest into two fragments. They move in opposite directions with different velocities. Here the momentum is conserved and the centre of mass of decay product continuous to be at rest.

2) **Motion of binary stars**: In case of motion of binary stars, if no external forces act, the centre of mass moves like a free particle.

Centre of mass of a rigid body

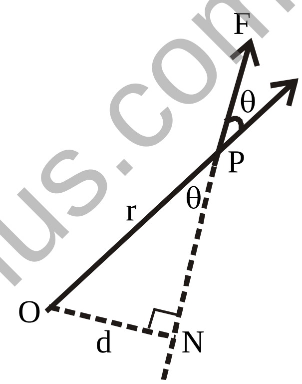
The center of mass of a rigid body is a fixed point with respect to the body as a whole. Depending on its shape and mass distribution, the center of mass of a rigid body may or may not be a point within the body.

Centre of mass of some regular bodies.

A uniform rod – at the geometric center.
A ring or a uniform disc – at the center.
A uniform cylinder – at the center of its axis of symmetry.
A triangle – at the point of intersection of the medians.

Torque

*The torque or moment of force about a point is the **turning effect of force about that point** and is measured as the product of force and the perpendicular distance between the point and the line of action of the force.*



Consider a particle P, whose position vector w r t origin O is $\vec{r}$.

Let a force $\vec{F}$ act on the particle in a direction making an angle θ with the direction of $\vec{r}$. Then torque, t = F x ON = F x d.

$$t = F \times r \sin\theta \qquad \left[\because \sin\theta = \frac{d}{r}\right]$$

$$\therefore \text{ torque, } \vec{\tau} = \vec{r} \times \vec{F}$$

Here **d is called moment arm** of the force.

The direction of torque vector is normal to the plane containing and .

** Unit of torque = newton metre (N m)

Component form of torque

Since $\vec{r}$ and $\vec{F}$ lie in X – Y plane, and $\vec{r} = \hat{i}x + \hat{j}y$ and $\vec{F} = \hat{i}\vec{F}_x + \hat{j}\vec{F}_y$

Now, torque, $\vec{\tau} = \vec{r} \times \vec{F} = (\hat{i}X + \hat{j}Y) \times (\hat{i}\vec{F}_x + \hat{j}\vec{F}_y) = K[xF_y - yF_x]$

Angular Momentum

The angular momentum is the moment of momentum of the particle about a point. It is the product of momentum of the particle and perpendicular distance between the point and direction of momentum.

Consider a particle P whose position vector w r t origin O is . Let $\vec{p} = m\vec{v}$ be the momentum of the particle, which makes an angle θ with $\vec{r}$.

Then angular momentum, L = p x d.

$$L = p \times r \sin\theta$$

$$\therefore \text{ Angular momentum, } \vec{L} = \vec{r} \times \vec{p}$$

Its direction is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

Also, angular momentum, $\vec{L} = \vec{r} \times \vec{p}$.

Differentiating wrt time, $\frac{dL}{dt} = \frac{d}{dt}(r \times p) = \frac{dr}{dt} \times p + r \times \frac{dp}{dt}$

But $\frac{dr}{dt} \times p = v \times mv = 0 \;\{\because v \times v = 0\}$

$$\therefore \frac{dL}{dt} = r \times \frac{dp}{dt} = r \times F = \tau \text{ (torque)}$$

i.e. $\frac{dL}{dt} = \tau$ i.e Rate of change of angular momentum is equal to the torque acting on the body.

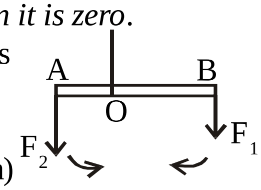
Couple

A pair of equal and opposite forces with different line of action is known as a couple or torque.

Condition for rotational equilibrium.

A body is said to in rotational equilibrium, when the net torque acting on it is zero.

Consider a rigid rod suspended by means of a string from a rigid support. Two forces F_1 and F_2 acts vertically downwards at the end of the rod as shown. The force F_1 produces a torque or moment of force = F_1 x OB and tries to rotate the rod in the clockwise direction. (If there is F_1 alone, the rod will rotate along clockwise direction)



The force F_2 produces a torque = F_2 x OA and tries to rotate the rod in the anti clockwise direction. When these torques or moments of these forces are same, the rod is in equilibrium and remains horizontal. In that case,

$$F_2 \times OA = F_1 \times OB.$$

Or *in rotational equilibrium, total anti clockwise moment = total clockwise moments. This is called* **principle of moments.**

The forces can be applied in the form of weights by hanging masses m_1 and m_2 at the ends. In that case, $F_1 = m_1g$ and $F_2 = m_2 g$ and in equilibrium, F_1 x OB = F_2 x OA.

i.e. $m_1g \times OB = m_2g \times OA$.

Cancelling g on both sides,

$$m_1 \times OB = m_2 \times OA.$$

Knowing the values of m_2, measuring the values of OA and OB, we can calculate the value of m_1 using the formula $m_1 = \frac{m_2 \times OA}{OB}$. when the rod is exactly horizontal.

Note: The point O is called *fulcrum*. One of the mass m_1 can be called the *load* and the mass m_2 suspended to raise m_1 can be called as *effort*. OB is called load arm and OA is called *effort arm*.

Moment of inertia (I).

Inertia in linear motion is the inability of a body to change its state of rest or of uniform motion in a straight line, without the help of an external force. A corresponding property of the body in rotational motion is known as moment of inertia.

" Moment of inertia of a body about a given axis is defined as the property of the body by virtue of which it is unable to change its position of rest or of uniform rotational motion without the help of external torque".

Moment of inertia depends on two factors: 1) mass of the body. 2) distribution of mass about the axis of rotation.

Moment of inertia of a particle.

Moment of inertia of a particle about an axis is the product of the mass of the particle and the square of the distance of the particle from the axis.

If a particle of mass m is placed at a distance r from the axis of rotation, then moment of inertia, $\mathbf{I = mr^2}$.

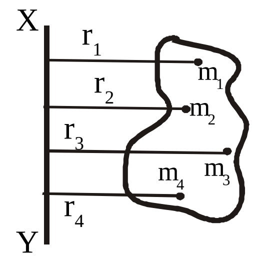
Unit : kg m^2. Dimension : ML^2.

Moment of inertia of a rigid body

Consider a rigid body capable of rotating about an axis XY. Let m_1, m_2, m_3 ,….. be the masses of various particles situated at distances r_1, r_2, r_3,……. from axis XY.

Then moment of inertia of the rigid body about XY,

$$I = m_1r_1^2 + m_2r_2^2 + \ldots\ldots = \sum mr^2$$



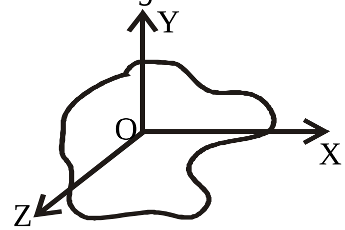
Radius of Gyration (K)

It is the distance between centre of mass and axis of rotation. $I = MK^2 \quad \therefore \quad K = \sqrt{\frac{I}{M}}$ Unit: metre.

NB **Theorems of Moment of Inertia.

** 1) **Perpendicular axes theorem.**

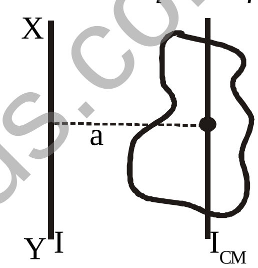
It states that " *the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moment of inertia about two mutually perpendicular axes lying in its plane and intersecting each other at the point where the perpendicular axis passes through the lamina*".



If I_X and I_Y are the moments of inertia of the lamina about $\perp^r$ axes OX and OY in the plane of lamina, then moment of inertia of lamina about Z axis (OZ) , $I_Z = I_X + I_Y$.

** 2) **Parallel axes theorem**

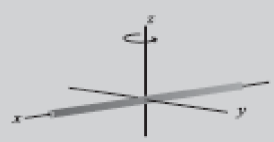
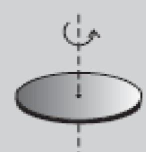
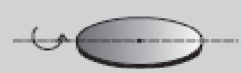
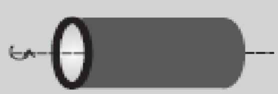
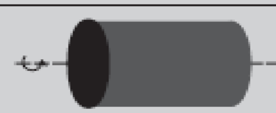
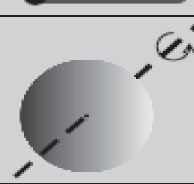
"*Moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis through its center of mass and the product its mass and square of the distance between the two axes*".



If I_{CM} is the moment of inertia of the body about an axis passing through the center of mass, then the moment of inertia of the body about a parallel axis distant **a** from the axis through center of mass,

$I = I_{CM} + M a^2$; where M is the mass of the body.

The moment of inertia of some rigid bodies are given in the table below.

Z	Body	Axis	Figure	I
(1)	Thin circular ring, radius R	Perpendicular to plane, at centre		$M R^2$
(2)	Thin circular ring, radius R	Diameter		$M R^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$M L^2/12$
(4)	Circular disc, radius R	Perpendicular to disc at centre		$M R^2/2$
(5)	Circular disc, radius R	Diameter		$M R^2/4$
(6)	Hollow cylinder, radius R	Axis of cylinder		$M R^2$
(7)	Solid cylinder, radius R	Axis of cylinder		$M R^2/2$
(8)	Solid sphere, radius R	Diameter		$2 M R^2/5$

Work and power in rotational motion.

Consider a force F acting at the rim of a pivoted wheel of radius R. During the action, the wheel rotates through a small angle $d\theta$. If this angle is small enough, the direction of force remains constant. Let the corresponding displacement is dx within a time interval dt.

Then work done, $dw = F\,dx = F\,R\,d\theta$. {Arc length = angle x radius}

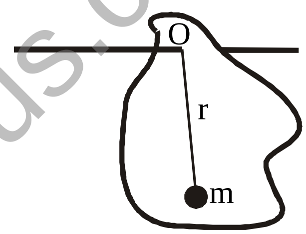
But F R is the torque τ.

$\therefore$ workdone, $dW = \tau\, d\theta$. and work done to turn from θ_1 to θ_2 is given by $W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$

Power, $P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$ where ω is the angular speed.

$\therefore$ Work done by torque for a small angular displacement $d\theta$ is given by $\mathbf{dW = \tau\, d\theta}$. and Instantaneous power, $\mathbf{P = \tau\,\omega}$.

** NB **Kinetic energy of a rotating body**



Consider a rigid body rotating about an axis passing through O with uniform angular velocity ω. This body can be considered to be made up- of a large number of particles. Consider one such particle of mass m at a distance r from O.

Then linear velocity of particle, $v = r\,\omega$.

KE of particle $= \frac{1}{2} mv^2 = \frac{1}{2} m\, r^2\omega^2$.

KE of whole body $= \sum \frac{1}{2} m\, r^2\omega^2 = \frac{1}{2}\omega^2 \sum m\, r^2 = \frac{1}{2} I\omega^2$.

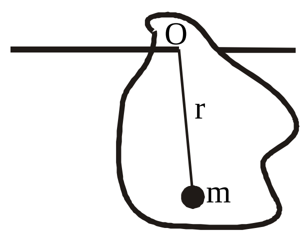
Where $\sum m\, r^2 = I$, the MI of body about the axis.

$\therefore$ **KE of rotating body** $= \frac{1}{2} I\omega^2$.

NB **Angular momentum of a rotating body

The sum of the moments of linear momentum of all the particles of the body about the axis of rotation is called its angular momentum about that axis.

Consider a body rotating about an axis. This body is made up of a large number of particles. Let one such particle of mass m be situated at a distance r from the axis.



Then linear velocity of particle, $v = r\,\omega$.

Linear momentum of particle $= m\,v = m\,r\,\omega$.

Moment of linear momentum about the axis $= m\,r\,\omega\, r = m\,r^2\omega$.

$\therefore$ Total moment of momentum of the whole body about the axis $= \sum m\,r^2\omega = I\omega$.

$\therefore$ **Angular momentum, L = Iω.**

Torque acting on a rigid body

Consider a rigid body rotating about a fixed axis with angular velocity ω. Let an external force act on the body. As a result, angular velocity of body changes. MI will remain constant and angular momentum changes. The rate of change of angular momentum gives the total external torque acting on the body.

ie torque, $\tau = \frac{dL}{dt} = I\frac{d\omega}{dt}$

or $\boldsymbol{\tau = I\,\alpha}$ $\quad \alpha = \frac{d\omega}{dt}$ = angular acceleration.

Law of conservation of angular momentum

When no external torque acts on a system of particles, the angular momentum of the system remains constant.

We know, torque $\tau = \frac{dL}{dt}$

If $\tau = 0, \quad \frac{dL}{dt} = 0.$

Then L = constant.

ie $\quad I\omega$ = constant $\quad$ Or $\quad I_1\omega_1 = I_2\omega_2.$

Examples

(1) An ice skater or a ballet dancer can increase his angular velocity by drawing his arms close to his body and bringing his stretched legs close together. (i.e, when the body is close together, $I = m R^2$ decreases so ω increases)

(2) A diver after leaving the spring board, curls his body by rolling the arms and legs inwards, so that I decreases and w increases. As he is about to touch the water surface, he stretches out his limbs so I increases and the diver enters water at a gentle speed.

Kinetic energy of rolling motion.

Consider a wheel rolling over a horizontal smooth surface without slipping This wheel has two types of motion: - (i) Rotational motion about an axis passing through its centre; and (ii) Linear motion in the horizontal direction.

Therefore, total KE = Translational KE + Rotational KE

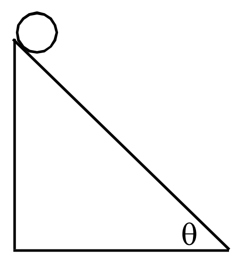
$$E = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

But $I = m K^2$ and $v = R\,\omega$.

$$\therefore \quad E = \frac{1}{2} m v^2 + \frac{1}{2} m K^2 \frac{v^2}{R^2} \qquad \text{i.e.} \quad E = \frac{1}{2} m v^2 \left(1 + \frac{K^2}{R^2}\right)$$

Note:

Acceleration of a body rolling down an inclined plane under gravity without slipping.



$$a = \frac{g \sin\theta}{\left[1 + \frac{I}{m R^2}\right]}$$

θ is the angle and I the moment of inertia.

$a_{ring} = \frac{1}{2} g \sin\theta \qquad a_{disc} = \frac{2}{3} g \sin\theta \qquad a_{hollow\ sphere} = \frac{3}{5} g \sin\theta \qquad a_{solid\ sphere} = \frac{5}{7} g \sin\theta$

Analogy between translational and rotational motion.

Translational motion	Rotational motion	Relation between them.
1. Linear displacement dx	1. Angular displacement $d\theta$.	1. $dx = r\, d\theta$.
2. Linear velocity $v = dx/dt$	2. Angular velocity $\omega = d\theta/dt$	2. $v = r\omega$.
3. Linear acceleration $a = dv/dt$.	3. Angular acceleration, $\alpha = d\omega/dt$	3. $a = r\alpha$.
4. Equations of motion under uniform acceleration. (i) $v = u + a\,t$. (ii) $S = ut + 1/2\, at^2$. (iii) $v^2 = u^2 + 2\,a\,S$.	4. Equations of motion under uniformly accelerated rotatory motion	
5. Mass M	5. Moment of inertia.	5. $I = M r^2$.
6. Force, F = Ma	6. Torque $\tau = I\alpha$	6. $\tau = r F \sin\theta$
7. Momentum p = mv.	7. Angular momentum $L = I\omega$.	7. $L = r\, p \sin\theta$.
8. Kinetic energy = $1/2\, mv^2$	8. Kinetic energy = $1/2\, I\, \omega^2$.	
9. Work done, dW = F dx (Force is in the direction of displacement.)	9. Work done $dW = \tau\, d\theta$	
10. Power P = F v	10. Power $= \tau\omega$	

MECHANICAL PROPERTIES OF SOLIDS

As we studied in the last chapter, a rigid body generally means a hard solid object having a definite shape and size. When suitable forces are applied on a body, it undergoes a change in length, volume or shape. Thus a wire fixed at its upper end pulled down by a weight at its lower end, undergoes a change in length. A solid body under compression undergoes a change in shape or change in volume etc.

Deforming force

If an external force is applied on a body, it may suffer a change in length or change in volume or change in shape. Such a body is said to be **deformed**. The applied force is called deforming force.

Thus the force required to change the length, volume or shape of a body is called deforming force.

Elasticity

When the deforming forces are removed, the body shows a tendency to recover its original condition.

"The property of a material body by which it regains its original dimensions (length, volume, shape) on the removal of the deforming forces is called elasticity".

Bodies which can completely recover their original condition on the withdrawal of the deforming forces are called **perfectly elastic** (Quartz fibre is the nearest approach to perfectly elastic).

Plasticity

Some bodies do not show any tendency to recover their original conditions They are said to be plastic and this property is known as plasticity. (Paraffin wax, putty, kneaded flour etc are close to ideal plastics)

There is a maximum value for the deforming force, beyond which the body ceases to be elastic. *This maximum value of the deforming force is called* **elastic limit** *of the body.*

Stress & Strain

Stress

When the deforming force is applied to a body, internal forces are set up in the body which opposes the deformation and would tend to restore the body back to its original condition.

Restoring force setup inside the body per unit area is called **stress**.

$$\therefore \quad \text{stress} = \frac{\text{restoring force}}{\text{area}}$$

As the restoring force set up in the body is equal and opposite to the external deforming force (within elastic limits), the stress may be measured as the external force acting per unit area.

i.e $\text{stress} = \dfrac{\text{external applied force}}{\text{area}} = \dfrac{F}{a}$. Unit: N/m^2. or pascal (Pa). Dimension $ML^{-1}T^{-2}$.

There are three ways in which a solid may change its dimensions when an external force acts on it.

When a wire (cylinder) is stretched by two equal **forces applied normal to its cross sectional area**, the restoring force per unit area is called **tensile stress**.

If the body is compressed under the action of applied forces the restoring force per unit area is called **compressive stress**.

Tensile or compressive stress can also be termed as **longitudinal stress or normal stress**.

If the deforming force applied tangential to a body (i.e. parallel to its cross sectional area), then the restoring force per unit area developed due to this tangential force is known **tangential or shearing stress**.

Strain

When a deforming force acts on a body, it undergoes change in its dimensions and the body is said to be deformed or strained.

The ratio of the change in dimension of a body to the original dimension is called strain.

$$\text{Strain} = \frac{\text{change in dimension}}{\text{original dimension}}$$

It has no unit as it is the ratio of two similar quantities.

There are three kinds of strains depending upon the nature of the deformation produced.

Linear or longitudinal strain

If the deforming force produces change in the length of the body, then the strain is known as linear or longitudinal strain.

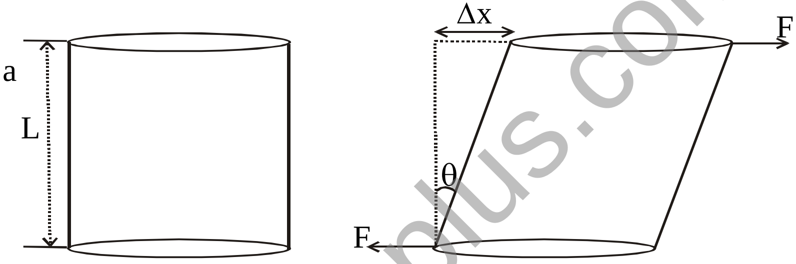
$$\text{Longitudinal strain} = \frac{\text{change in length}}{\text{original length}}$$

If Δ L is the increase in length of a wire of original length L, then $\text{Longitudinal strain} = \frac{\Delta L}{L}$

Volume strain $\text{volume strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\Delta V}{V}$

Shearing strain

Consider a cylinder of length L. Let a tangential force be applied so that it undergoes a relative displacement Δ x. The strain so produced is knownas *shearing strain and is*



defined as $\text{shearing strain} = \frac{\Delta x}{L} = \tan\theta$; where θ is the angular displacement of the cylinder from the vertical (original position of cylinder). Since θ is very small, tanθ is nearly equal to the angle θ. So

$$\theta = \frac{\Delta x}{L}$$

Hooke's law

Within elastic limit, stress is directly proportional to strain.

$$\text{stress} \propto \text{strain}$$

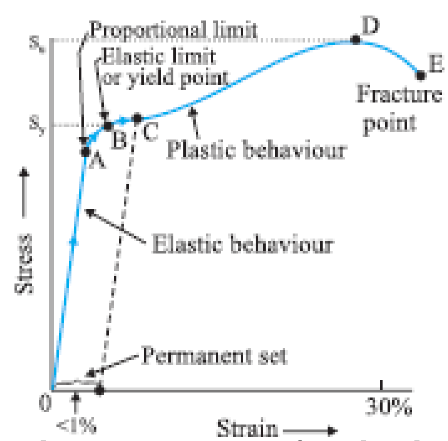
$$\text{stress} = K \times \text{strain}$$

where K is the proportionality constant and is known as **modulus of elasticity**.

Stress - strain curve.

The relation between stress and strain for a given material under tensile stress can be found experimentally. Let a wire or cylinder is stretched by an applied force. The applied force is gradually increased and the change in length is noted. A graph is plotted between stress and the strain produced. A typical graph for a metal is shown in fig.

The stress - strain curves vary from material to material. From the graph, we can that in the region between O to A, the curve is linear. In this region, Hooke's law is obeyed.



A typical stress - strain curve for a ductile metal.

The body regains its original dimension when the applied force is removed. In this region, the solid behaves as an elastic body.

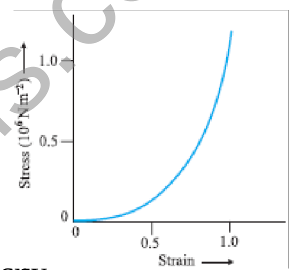
In the region from A to B, stress and strain are notproportional, but the body returns to its original dimension when the load is removed.The point A is called **proportional limit**. The point B in the curve is known as **yield point (elastic limit)**. The corresponding stress is known as **yield strength** (S_y) of the material.

If the load is increased further, the stress developed exceeds the yield strength and strain increases rapidly even for a small change in the stress. The portion of the curve between B and D shows this.

When the load is removed, say at some point C between B and D, the body does not regain its original dimension. In this case, even when the stress is zero, the strain is not equal to zero. The material is said to have a **permanent set**.
The deformation is said to be **plastic deformation**. The point D on the graph is the ultimate tensile strength (S_u) of the material. Beyond this point, additional strain is produced even by a reduced applied force and fracture occurs at point E. E is known as **fracture point**.

Materials which break as soon as the stress is increased beyond the elastic limit are called **brittle** (Eg: cast iron)
Materials which have a good plastic range are called ductile (Eg: Cu, Ag, Al etc) Such materials can be used for making springs, sheets etc.
Materials like rubber and elastic tissue of aorta exhibit a different stress - strain relationship. By applying a relatively small stress, the length of rubber can be increases to several times its original length. On removal of the stress, it returns to its original length. Thus rubber has a large elastic region. But it has no well defined plastic region. They don't obey Hooke's law as the stress - strain graph is not a straight line.

Stress - strain curve for the elastic tissue of Aorta or rubber is as shown.
Substances that can be elastically stretched to large values of strain are called **elastomers**.



Elastic moduli (Modulus of elasticity)

According to Hooke's law, within elastic limit,

$$\text{stress} \propto \text{strain} \quad \text{or} \quad \frac{\text{stress}}{\text{strain}} = K$$ where K is known as modulus of elasticity.

There are three types of moduli - Young's modulus, Bulk modulus and Shear modulus or Rigidity modulus according to the deformation in length, volume or shape respectively. Unit of modulus of elasticity is N/m^2 or pascal (same as that of stress)

Young's modulus (Y)

The ratio of tensile or longitudinal stress to the longitudinal strain is called Young's modulus.

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$\text{longitudinal stress} = \frac{F}{A} \qquad \text{longitudinal strain} = \frac{\Delta L}{L} \qquad \therefore \quad Y = \frac{F/A}{\Delta L/L}$$

OR $$Y = \frac{FL}{A\,\Delta L}$$

Now if F = mg. and r is the radius of cross section of wire, then area $A = \pi r^2$.

Therefore, $$Y = \frac{mgL}{\pi r^2\,\Delta L}$$

Note: It is noticed that for metals, Young's moduli are large. That is these materials require large force to produce mall change in length.

It is also noticed that the force required to produce a strain in a steel wire is greater than that required to produce the same strain in aluminium, brass or copper wires of same cross-sectional area. It is for this reason that steel is preferred in heavy - duty machines and in structural designs. Wood, bone concrete and in glass have rather small Young's moduli.

Bulk modulus

Within elastic limit, the ratio of the normal stress (hydraulic stress) to the volume strain (hydraulic strain) is called bulk modulus of the material.

Let F be the force applied normally and uniformly on a body of surface area a, then normal stress $= \frac{F}{a} = P$, the change in pressure. Let V be the original volume of the body and ΔV be the change in volume produced by the stress P.

$$\text{Then, volume strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{-\Delta V}{V}$$

The -ve sign shows that on increasing the pressure, volume decreases.

$$\text{Bulk modulus, } B = \frac{-P}{\Delta V / V} = \frac{-PV}{\Delta V}$$ Unit is Pa or Nm^{-2}.

B of solids are much higher than liquids and very much higher than that of gases.

Compressibility (K)

The reciprocal of bulk modulus of the material is known as compressibility.

Compressibility, $K = \frac{1}{B} = \frac{-\Delta V}{PV}$ Its unit is $N^{-1}m^2$. or Pa^{-1}.

Shear modulus or Rigidity modulus

The ratio of shearing stress to corresponding shearing strain is called the shear modulus (G).

$$G = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{F/A}{\Delta x / L} = \frac{FL}{A\,\Delta x}$$

OR, we know that $\text{shearing strain} = \frac{\Delta x}{L} \approx \theta$. So, $G = \frac{F}{A\theta}$. Unit Nm^{-2} or Pa.

Applications of elastic behavior of materials.

Most of the materials used by us and in nature are under some kind of stress. It is therefore important to design things in such a way that they can continue to function under the stress acting on them.

One of the most important applications of elasticity is **to find out the thickness required for a metal rope used in cranes to pull up heavy load**.

Let it be required to raise a load of 10 tons. (1 ton - 1000 kg) using a steel rope. The rope must be such that the extension should not exceed its elastic limit. The elastic limit of steel corresponds to a stress of 3 x 10^8 Nm^{-2}.

$$\text{stress} = \frac{F}{a} \quad \therefore a = \frac{F}{\text{stress}}$$

$F = mg = 10 \times 10^3 \times 9.8 \approx 10^5$ N.

Stress $\leq 3 \times 10^8\ Nm^{-2}$.

$$\text{area} \geq \frac{10^5}{3 \times 10^8} \geq 3.3 \times 10^{-4} \text{ m.}$$

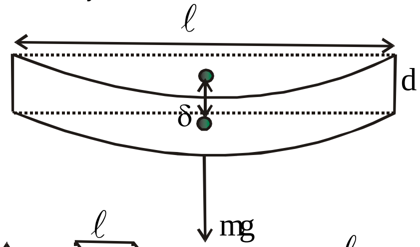
$\pi r^2 \geq 3.3 \times 10^{-4}$ m. $\therefore r^2 \geq 10^{-4}$ m. Or $\therefore r \geq 10^{-2}$ m or 1cm. (approximately).

i.e. the radius of the rope must be greater than 1 cm. In practice, the rope is made of a large number of thin wires braided together.

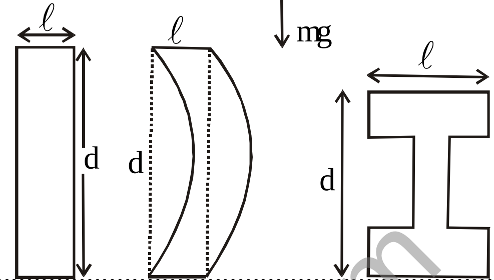
Another application of elasticity is in t**he design of bridges for maximum safety**.

A bridge should be so designed that it should not bend too much or break under its own weight or under the load of traffic. The depression, δ at the centre of a beam of length, ℓ, breadth b and thickness d under a load mg at its midpoint is given by, $\delta = \frac{mg\ell^3}{4bd^3Y}$.

Hence to reduce the bending, for a given load, Y of the material should be large, b and d of the beam must also be large and ℓ should be as small as possible. Since δ is inversely proportional to d^3, the depression can be reduced more effectively by increasing the thickness d rather than increasing breadth b of the beam.



But on increasing the thickness, unless the load is at the centre, the beam bend as in fig. This is called '**buckling**' of the beam. To prevent buckling, a large load bearing surface is required. Hence the beam is designed to have a large thickness to minimise 'bending' and a large load bearing surface to prevent 'buckling'. It is called **I section** of the beam.



MECHANICAL PROPERTIES OF FLUIDS

Unlike solids, liquids or gases do not possess definite shape. Further, they flow under the action of forces, hence the name fluids.

Pressure

When an object is submerged in a fluid at rest, the fluid exerts a force on the surface. This force is always normal to the object surface.

If F is the magnitude of the normal force on an area A, then the average pressure P_{av} is defined as the normal force acting per unit area. $P_{av} = \frac{F}{A}$.   Pressure is a scalar quantity.

{Note: It is the component of the force normal to the area under consideration and not the (vector) force that appears in the above equation}

Its dimension is $ML^{-1}T^{-2}$ and unit Nm^{-2} or pascal (Pa). Another common unit of pressure is atmosphere (atm).

i.e the pressure exerted by the atmosphere at sea level. (1atm = 1.013×10^5 Pa)

Density

For a fluid of mass m occupying a volume, V, the density $\rho = \frac{m}{V}$. Dimension ML^{-3}. Unit kg m^{-3}. It is a +ve scalar quantity.

A liquid is largely incompressible and its density is therefore, nearly constant at all pressures. Gases on the other hand, exhibit large variations in densities with pressure. The density of water at 4^0C is 1×10^3 kgm^{-3}.

The ***relative density*** *of a substance is the ratio of its density to the density of water at* 4^0C .

Pascal's law

It states that *"the pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and to the walls of the containing vessel"*.

That is the pressure at any point of a confined fluid, the pressure at every other point of the fluid is also changed by the same extend or the fluid transmits pressure equally in all directions.

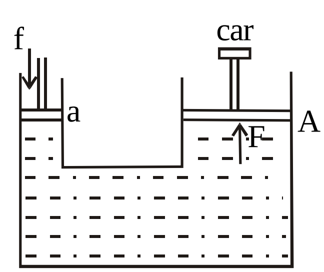
Hydraulic machines

A number of devices such as hydraulic lift and hydraulic brakes are based on the Pascal's law.

Hydraulic lift

It is use to support or lift heavy objects. The principle used in it is Pascal's law.

It consists of two cylinders of different areas of cross-section and are provided with light friction less pistons. They are connected to each other by means of pipe. The load to be lifted is placed on the platform attached with the larger piston. The space below the piston is filled with a liquid.



Let the force 'f' be applied on the smaller piston of cross-sectional area a.

Then the pressure exerted on the liquid = $\frac{f}{a}$.

According to pascal's law, this pressure is transmitted through the liquid to the larger piston of cross-sectional area A. Then force 'F' exerted on the larger piston is given by

$$F = \text{pressure x area} = \frac{f}{a} \text{ x } A$$

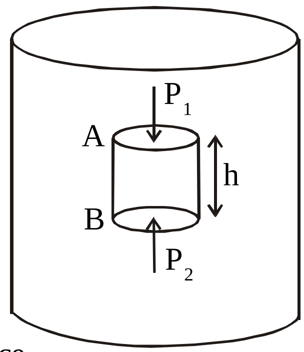
i.e. $F = \frac{A}{a} f$. Since A is much larger than a, the force F on the larger piston will be very much greater than the applied force 'f'. This force can be used to lift a heavy weight, say a car placed on the platform.

Thus the hydraulic lift is essentially a force multiplying device, the multiplication factor is equal to the ratio of areas of the pistons $\left[\frac{A}{a}\right]$

Hydraulic brakes

It is also based on Pascal's law. When we apply a small force on the pedal with our foot, the master piston moves inside the master cylinder and the pressure caused is transmitted through the brake oil to act on a piston of larger area. A large force acts on the piston and is pushed down expanding the brake shoes against brake lining. In this way a small force on the pedal produces a large retarding force on the wheel. An important advantage of this system is that the pressure set up by pressing pedal is transmitted equally to all cylinders attached to the four wheels so that the braking effort is equal on all wheels.

Variation of pressure with depth.



Consider a fluid of density ρ contained in a vessel. To find the pressure difference between two points A and B at a vertical distance h, consider an imaginary cylinder of the liquid (fluid) of cross-sectional area a, such that the point A and B lie on the upper and lower circular faces of the cylinder.

Volume of the cylindrical column of fluid = ah.
mass of the fluid = V ρ = ahρ.
Weight of the fluid = ahρg ; where g is the acceleration due to gravity at the place.

The weight of the fluid column acts vertically down wards. Let P_1 and P_2 be the fluid pressures at the points A and B respectively.

Force on the upper face of the cylinder acting vertically downwards, $F_1 = P_1 a$.
Force on the lower face of the cylinder acting vertically upwards, $F_2 = P_2 a$.
Forces on the cylindrical surfaces cancel each other. Since the imaginary cylinder of fluid is in equilibrium, the resultant force on it must be zero.

i.e. Total upward force = Total downward force

$$P_2 a = P_1 a + ah\rho g$$
$$(P_2 - P_1)a = ah\rho g$$
$$\therefore \quad P_2 - P_1 = h\rho g$$

Pressure difference depends on the vertical distance 'h' between the points, density of the fluid, acceleration due to gravity at the place g.

If A lies on the free surface of a liquid, $P_1 = P_a$, the atmospheric pressure and $P_2 = P$. Then

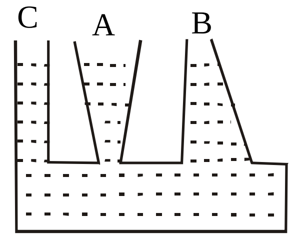
$P - P_a = h\rho g$ OR $P = P_a + h\rho g$

Thus pressure at a depth below the surface of a liquid open to the atmosphere is greater than atmospheric pressure by an amount hρg. The excess of pressure $P - P_a$ at a depth is called a **gauge pressure** at that point.

The equation for pressure shows that the pressure exerted by a liquid at a point depends only on the depth of the point and the density of the liquid. It doesn't depend on the shape of the container. Hence *the pressure at all points on the same horizontal level at rest is the same.*

Hydrostatic paradox

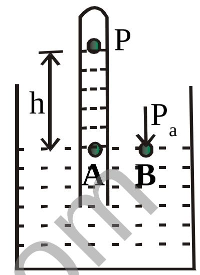
Consider three vessels A, B and C of different shapes and volumes. They are connected at the bottom by a horizontal pipe. On filling water, the level in the three vessels is the same though they hold different amounts of water. As the pressure depends only on the depth below the free surface and not on the shape of the containing vessel, the pressure at the base of all these vessels is the same and hence the system is in equilibrium.



Atmospheric pressure

The pressure exerted by the atmosphere at any point is equal to the weight of the column of air of unit area of cross-section extending from that point to the tip of the atmosphere. At sea level it is 1.013 x 10^5 Pa. (1atm)

The first experimental method of measuring the atmospheric pressure was designed by the Italian scientist E. Torricelli. He filled a long strong glass tube closed at one end with mercury and inverted it into a trough of mercury. This device is known as **mercury barometer**.



The space above the mercury column in the tube contains only mercury vapor whose pressure P is so small that it may be neglected. The pressure inside the column at a point A must be equal to the pressure at point B which is at the same level.

Pressure at B = atmospheric pressure = P_a. $P_a = h\rho g$; where ρ is the density of mercury, h is the height of mercury column in the tube. It is found that the mercury column in the barometer has a height of about 76 cm at sea level - equivalent of one atmosphere (1atm).

So 1 atm = 76 cm of Hg.

Other units of atmospheric pressure are 1 torr = 133 Pa = 1 mm of Hg. 1 bar = 10^5 Pa.

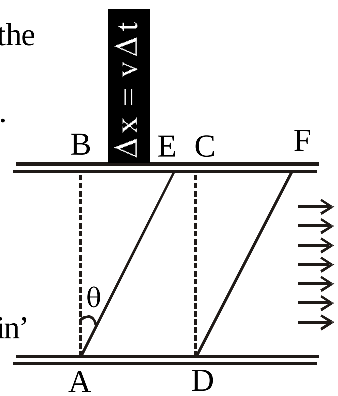
Viscosity

Let us pour equal amounts of water and honey in two similar funnels. Water flows out the funnel very quickly but honey is extremely slow in flowing down. This indicates that *there must be some internal frictional force which opposes the relative motion between different layers of the liquid. This force is called* **viscous force**.

"Viscosity is the property of a fluid by virtue of which it opposes relative motion between its different layers".

Liquids are more viscous than gases. The greater the viscosity, the lesser is the fluid flow.

Consider a portion of liquid which at some instant has the shape ABCD. Take the shape of ABCD after a short interval of time Δ t. During this time interval, the liquid has undergone a shear strain of $\frac{\Delta x}{\ell}$.



The strain in a flowing fluid increases with time continuously. Unlike a solid, here the stress is found experimentally to depend on the 'rate of change of strain' or 'strain rate' i.e. $\frac{\Delta x}{\ell \Delta t} = \frac{v}{\ell}$; instead of strain.

The coefficient of viscosity η (pronounced as eta) for a fluid is defined as the ratio of shearing stress to the strain rate i.e $\eta = \frac{F/A}{v/\ell} = \frac{F\ell}{vA}$ The SI unit of viscosity is **poiseiulle** (PI) OR Nsm^{-2} or PaS.

Dimension $ML^{-1}T^{-1}$.

The viscosity of liquids decrease with temperature while it increases in the case of gases.

Stoke's law

When a body falls through a fluid, it drags the layer of the fluid in contact with it. A relative motion between different layers of the fluid is set and as a result, the body experiences a retarding force. It is seen that the viscous force is proportional to the velocity of the object and is opposite to the direction of motion. The other quantities on which the force F depends are viscosity η of the fluid and radius a of the sphere.

An English scientist George G Stokes showed that the viscous drag force, **F = 6 π ηa v. This is known as Stoke's law.**

Consider a lead shot falling in caster oil contained in a measuring jar. It accelerates initially due to gravity.But the viscous force of caster oil provides a retarding force for the lead shot. As velocity increases, the retarding force also increases. Finally when viscous force plus buoyant force becomes equal to force due to gravity (weight), the net force and hence acceleration becomes zero. The lead shot, then descends with a constant velocity known as **terminal velocity** v_t.

. *Terminal velocity of a body is defined as the constant maximum velocity acquired by a body while falling through a viscous fluid.*

So $6\pi\eta a v_t +$ up thrust = weight of body.

Upthrust on the sphere = weight of the fluid displaced = volume of sphere x density of fluid x g = $\frac{4}{3}\pi a^3 \times \sigma g$

Weight of the body = mg = volume of body x density of body x g. = $\frac{4}{3}\pi a^3 \times \rho g$

$$\text{So} \quad 6\pi\eta a v_t + \frac{4}{3}\pi a^3 \times \sigma g = \frac{4}{3}\pi a^3 \times \rho g$$

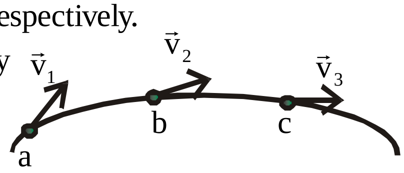
$$6\pi\eta a v_t = \frac{4}{3}\pi a^3 (\rho - \sigma) g \qquad \therefore \qquad v_t = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{\eta}$$

So terminal velocity v_t depends on the square of the radius of the sphere and inversely on the viscosity of the medium.

Streamline flow

If every particle of the fluid, follows the path of its preceding particle with exactly the same velocity, then the flow is a streamline flow of the fluid.

In the streamline flow of a liquid, the velocity of every particle crossing a particular point is the same. However, the velocities of the particles at different points in their path may or may not be the same. The path followed by any particle of the fluid in streamline flow is called a streamline. It may be straight or curved. Here $\vec{v}_1, \vec{v}_1$ and $\vec{v}_3$ are the velocities of the particles at the points a, b and c respectively.



(i)The tangent at any point of the streamline gives the direction of velocity of the liquid at that point.

(ii)Two streamlines cannot intersect. If they intersect, then there will be two different directions of velocity at a given point. That is impossible.

(iii)At a particular point of the streamline, the velocity of liquid is constant. However, at different points of the streamline, the velocity may be different.

Turbulent flow

The flow of a liquid is said to be turbulent or disorderly if its velocity is greater than a particular velocity known as critical velocity.

Critical velocity of a fluid is that velocity of fluid up to which its flow is streamline above which its flow becomes turbulent.

Eg: * A stream of fluid flowing past an obstacle changes this into turbulent

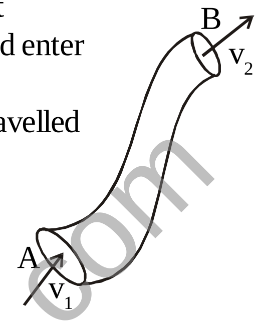
* The wave left in water by moving ships
* Rising smoke becomes turbulent after some time.
* The blades of kitchen mixers induces turbulent flow.

Reynold's number.

Reynolds number is a pure (dimensionless) number which determines the type of flow of a liquid through a pipe. It is denoted by R_e.

$R_e = \dfrac{\rho v d}{\eta}$; where ρ is the density of the fluid flowing with a speed v, d is the diameter of the pipe, η is the viscosity of the fluid. It is found that flow is streamline or laminar for R_e less than 1000. The flow is turbulent for $R_e > 2000$. The flow becomes unsteady for R_e between 1000 and 2000. So nature of flow depends upon velocity, density, dimensions of pipe and coefficient of viscosity.

Equation of continuity



Consider a streamline flow of a fluid through a pipe of varying cross - section. Let a_1 and a_2 be the cross-sectional area of the pipe at A and B respectively. Let the liquid enter the end A normally with velocity v_1 and leave the end B normally with velocity v_2.

Volume of fluid flowing per second through A = $a_1 v_1$ {volume = area x distance travelled in one second}

Mass of fluid flowing per second through A = $a_1 v_1 \rho_1$; where ρ_1 is the density of the fluid at A.

Similarly mass of fluid flowing per second through B = $a_2 v_2 \rho_2$; where ρ_2 is the density of the fluid at B.

Since no fluid is created or destroyed in the tube, the mass of fluid crossing each section of the tube per second must be the same.

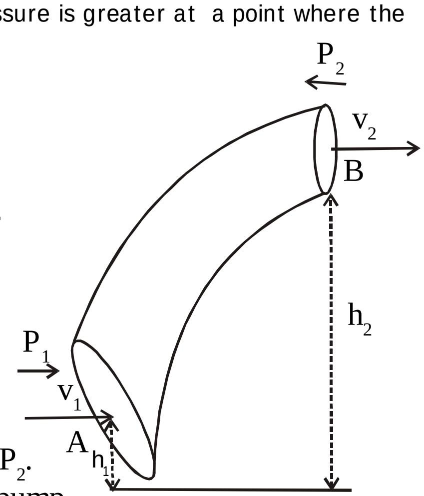
$\therefore a_1 v_1 \rho_1 = a_2 v_2 \rho_2$ i.e $a\,v\,\rho$ = constant

Since liquids are incompressible, $\rho_1 = \rho_2$. So $\mathbf{a_1 v_1 = a_2 v_2}$. i.e. the volume of liquid entering the section A per second is equal to that leaving the section B per second.

Or **a v = constant**. This is the equation of continuity for the streamline flow of an incompressible fluid. It is a statement of conservation of mass in flow of incompressible fluids.

* The velocity of flow of the liquid is inversely proportional to the cross-sectional area. The velocity of flow increases as the cross-sectional area decreases and vice versa.
The cross-sectional area available for flowing water goes on increasing as we approach the bottom of a river. So velocity of the flow decreases. This explains as to why deep water runs slow.
** Since the velocity of flow of the liquid increases from A to B, there must be some accelerating force. So the pressure at the end A must be greater than the pressure at the end B. So pressure is greater at a point where the velocity is small and vice-versa.

Bernoulli's Principle



Consider the streamline flow of an ideal fluid in a pipe of varying cross-sectional area A and B. Let the pipe be at varying heights. Let P_1 and P_2 be the pressures at the ends A and B. a_1 and a_2 be the area of cross-section. v_1 and v_2 are velocities. {Here the fluid is ideal. So no viscous force or frictional force acts on it and hence volume will be the same}.

Then from the equation of continuity, $a_1 v_1 = a_2 v_2$.

Here $a_2 < a_1$, so $v_2 > v_1$.

So the fluid accelerates against the force of gravity from A to B. Or $P_1 > P_2$.

This pressure difference is maintained by some external agency like a pump.

Let a very small mass m of the fluid enters the pipe in a time 't'. Then

distance covered in time t = $v_1 t$.

This distance is parallel to the axis of the tube and in the direction of the force $P_1 a_1$. Let W_1 be the *work done on the fluid* by the force $P_1 a_1$.

$W_1 = P_1 a_1 .\; v_1 t$

The same mass m of the fluid leaves the end B in the same time t.

Then distance covered = $v_2 t$

But the distance is covered against the force $P_2 a_2$.

Then W_2 is the *work done by the fluid* against the force $P_2 a_2$.

$W_2 = P_2 a_2\, v_2 t$

Let W be the net work done by the pressure forces in driving a mass m of the fluid.

$$W = W_1 - W_2 = P_1 a_1 v_1 t - P_2 a_2 v_2 t \quad ....(1)$$

This work done is used in giving KE and PE to the fluid element of mass m.

$$\text{Increase in KE of mass m of fluid} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = \frac{1}{2} m \left(v_2^2 - v_1^2\right)$$

$$\text{Increase in PE of mass m of fluid} = m g h_2 - m g h_1 = m g (h_2 - h_1)$$

$$\text{Total increase in mechanical energy of the mass m of fluid, } E = \frac{1}{2} m \left(v_2^2 - v_1^2\right) + m g (h_2 - h_1)$$

Applying work - energy conservation principle, $W = E$.

$$P_1 a_1 v_1 t - P_2 a_2 v_2 t = \frac{1}{2} m \left(v_2^2 - v_1^2\right) + m g (h_2 - h_1)$$

The fluid is assumed to be incompressible and there is no source or sink in the pipe.
Therefore volume of fluid entering A = volume of fluid leaving B.

i.e. $a_1 v_1 t = a_2 v_2 t = \frac{m}{\rho}$; where ρ is the density of liquid.

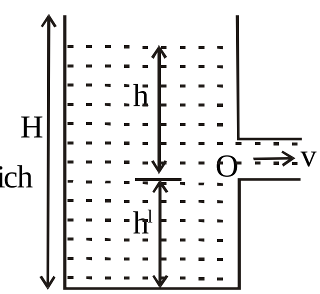
$$P_1 \frac{m}{\rho} - P_2 \frac{m}{\rho} = \frac{1}{2} m \left(v_2^2 - v_1^2\right) + m g (h_2 - h_1)$$

$$\text{i. e. } \frac{P_1}{\rho} - \frac{P_2}{\rho} = \frac{1}{2} \left(v_2^2 - v_1^2\right) + (h_2 - h_1) g$$

$$\frac{P_1}{\rho} + \frac{v_1^2}{2} + h_1 g = \frac{P_2}{\rho} + \frac{v_2^2}{2} + h_2 g \qquad \text{In general } \frac{P}{\rho} + \frac{v^2}{2} + g h = \text{constant}$$

Or $P + \frac{\rho v^2}{2} + h \rho g = \text{constant}$. This is **Bernoulli's theorem.**

"For the streamline flow of an ideal fluid, the sum of pressure (P), Kinetic energy per unit volume ($\frac{\rho v^2}{2}$) *and potential energy per unit volume* ($h \rho g$) *remains a constant".*



Applications of Bernoulli's theorem.

(i) Speed of efflux : Torricelli's law.

The word efflux means fluid out flow.

The velocity of efflux of a liquid through an orifice (small hole) is equal to that which a body would attain in falling freely from the free surface of a liquid to the orifice. This result was first obtained by Torricelli and is known as Torricelli's theorem or **law of efflux.**

Consider an ideal liquid of density ρ contained in a vessel having an orifice O. The vessel is assumed to be sufficiently wide and the opening O sufficiently narrow. So the velocity of the liquid on its free surface can be taken zero. The pressure at the free surface and on the orifice is atmospheric pressure. (P).

Let H and h^l be the height of the free surface of the liquid and the orifice. h be the height of the free surface of the liquid above the orifice. v be the velocity with which liquid flowing out through the orifice (i.e. velocity of efflux),

then applying Bernoulli's theorem,

Total energy per unit volume at the orifice = total energy per unit volume at the free surface.

$$P + h^l \rho g + \frac{\rho v^2}{2} = P + H \rho g + 0$$

$$\frac{1}{2}\rho v^2 = H\rho g - h^1\rho g = \rho g(H - h^1)$$

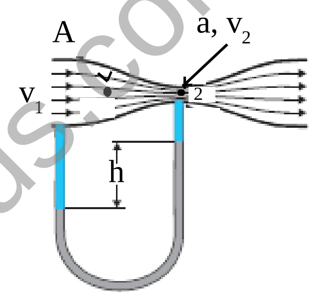
$$\frac{1}{2}\rho v^2 = h\rho g \quad \text{Or} \quad v = \sqrt{2gh} \quad ....(1)$$

Let a body fall freely from the level of the free surface of the liquid. Let v be the velocity acquired by the body when it reaches the level of orifice, then $v^2 - u^2 = 2aS$ i.e. $v^2 - 0 = 2gh$

$$\therefore \; v = \sqrt{2gh} \quad ....(2)$$

From (1) and (2), the velocity of efflux of a liquid is equal to the velocity acquired by the body in falling from the free surface of the liquid to the orifice. This proves Torricelli's theorem.

(ii) Venturimeter



The venturimeter is a device to measure the flow speed of incompressible fluid. It consists of a tube with broad diameter and a small constriction at the middle. A manometer in the form of a U-tube is attached to it, with one arm at the broad neck point of the tube and other at constriction.

From the equation of continuity, $av_2 = Av_1$. {Here $v_2 > v_1$ as $a < A$)

So speed at the cross-section becomes $v_2 = \frac{Av_1}{a}$

Then by Bernoulli's theorem, $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 = P_2 + \frac{1}{2}\rho\left[\frac{Av_1}{a}\right]^2$

So $P_1 - P_2 = \frac{1}{2}\rho v_1^2\left[\left(\frac{A}{a}\right)^2 - 1\right]$

This pressure difference causes the fluid in the U-tube connected at the narrow neck to rise in comparison to the other arm. The difference in height h of mercury in the U-tube gives the pressure difference.

So $P_1 - P_2 = \rho_m gh = \frac{1}{2}\rho v_1^2\left[\left(\frac{A}{a}\right)^2 - 1\right]$; where ρ_m is the density of mercury in the U-tube.

So speed of fluid at the wide neck is $v_1 = \sqrt{\frac{2\rho_m gh}{\rho\left[\left(\frac{A}{a}\right)^2 - 1\right]}}$

This principle behind this meter has many applications. The carburetor of automobile has a venturi channel (nozzle) through which air flows with a large speed. The pressure is then lowered at the narrow neck and the petrol is sucked up in the chamber to provide combustion. Filter pumps or aspirators, Bunsen burner, atomizers and sprayers used for perfumes work on the same principle.

(iii) Blood flow and heart attack

Bernoulli's principle helps in explaining blood flow in artery. The artery may get constricted due to the accumulation of plague on its inner walls. In order to drive the blood through this constriction, a greater demand is placed on the activity of the heart. The speed of flow of blood in this region is raised which lowers the pressure inside and the artery may collapse due to external pressure. The heart exerts further pressure to open this artery and forces the blood through. As the blood rushes through the opening, the internal pressure once again drops due to same reasons leading to repeat collapse. This may result in heart attack.

(iv) Dynamic lift

Dynamic lift is the force that acts on a body such as aeroplane wing, a hydrofoil or spinning ball by virtue of its motion through a fluid. In many games such as cricket, tennis, baseball or golf, we notice that a spinning ball deviates from its parabolic trajectory as it moves through air. This deviation can be partly explained on the basis of Bernoulli's principle.

(a) **Ball moving without spin:**

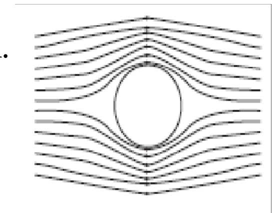
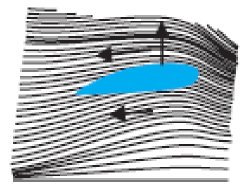


Figure shows the streamlines around a non-spinning ball moving relative to a fluid. Here the velocity of fluid above and below the ball at corresponding points is the same resulting in zero pressure difference. Therefore, no upward or downward force acts on the ball.

(b) **Ball moving with spin:**

A ball which is spinning drags air along with it. If the surface is rough, more air will be dragged. Fig shows the streamlines of air for a ball which is moving and spinning at the same time. The ball is moving forward and relative to it the air is moving backward. So the velocity of air above the ball relative to it is larger and below it is smaller. This difference i velocities of air causes a pressure difference between the lower and upper faces and there is a net upward force on the ball. This dynamic lift due to spinning is called ***Magnus effect***.

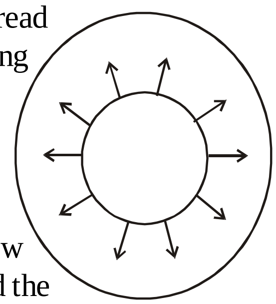
(c) **Aerofoil or lift on aircraft wing**.



Aerofoil is a solid piece shaped to provide an upward dynamic lift when it moves horizontally through air. The cross section of wings of an air craft looks some what like the aerofoil. When the aerofoil moves against the wing, the orientation of the wing relative to flow direction causes the streamlines to crowd together above the wing more than below it. The flow speed on top is higher than that below it. There is an upward force resulting in a dynamic lift of the wings and this balances the weight of the plane.

Surface tension

We have noticed that oil and water do not mix, mercury does not wet glass but water sticks to it, oil rises up a cotton wick in spite of gravity, hair of a paint brush do not cling together when dry but cling together when dipped in water. All these experiences are related with the free surfaces of liquids. These free surfaces possess some additional energy. This phenomenon is known as **surface tension.** It is concerned with only liquids as gases do not have free surfaces.



Consider a wire ring, a few centimeters in diameter has a loop of thread attached to it. A soap film is formed in this arrangement by dipping it in soap solution. The thread will remain on the film in any form. But if the film inside the loop is gently broken using a needle, then it spread out into a circle. The thread behaves as if the surface of the liquid where pulling it radially outwards. This pull every where is tangential to the surface of the film and has the same magnitude. *These forces are called the forces of surface tension or surface tension*.

Now for a given perimeter, a circle encloses the greatest area so that the loop now encloses the maximum area. That is the area of the soap film left behind the loop and the wire ring is now reduced to the minimum possible. The film behaves as if it were under a tension like a stretched elastic membrane. *This tension on the surface of a liquid is called the surface tension*.

Due to this tension, small drops of liquids assume spherical shape, since for a given volume, sphere has the least surface area.

Definition: *Surface tension is that property of a liquid by virtue of which it behaves like a stretched elastic membrane with a tendency to contract so as to have the minimum surface area*.

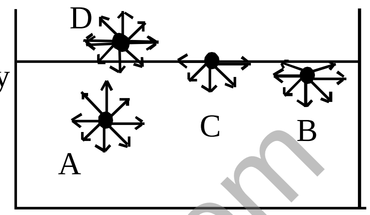
Surface tension of a liquid is measured as a force per unit length acting on either side of an imaginary line of the liquid surface, the direction of the force being tangential to the liquid surface and perpendicular to the line. Let F is the force acting on either side of a line of length ℓ. Then surface tension, $S = \frac{F}{\ell}$.

{If film (soap) then $S = \frac{F}{2\ell}$, as it has two surfaces} The unit of surface tension is N/m. Dimension MT^{-2}.

Explanation of surface tension based on molecular theory

Force of cohesion : - It is the force of attraction between molecules of same substance.
Force of adhesion: - It is the force of attraction between molecules of different substances.

A liquid consists of a very large number of molecules. These molecules attracts one another (cohesive force). *A sphere drawn with the molecules as centre and the range of molecular attraction as radius is called the sphere of influence of the molecule.*



XY represent the free surface of a liquid. Consider a molecule A well inside the liquid. This molecule is attracted equally from all directions by other molecules. So there is no resultant force acting on it. In the case of molecule B, the downward force of attraction will be greater than the upward force. Hence there is a resultant downward pull acting on it. The molecule C is just on the surface of the liquid. Therefore, the downward pull will be maximum. For the molecule D, the downward force of attraction is small, so the molecules can leave the liquid surface.

Thus the molecules on or very near to the surface experiences a resultant downward pull perpendicular to the surface, on account of the unbalanced forces of molecular attraction. Due to this downward pull, work must be done to transfer a molecule from the interior to the surface (against the force of cohesion). *This work done on the molecule is stored in it in the form of PE. So PE of a molecule lying on or near the surface is greater than that of a molecule well inside it. A system is in a stable equilibrium, when its PE is minimum.* For this the number of molecules in the surface of film should be minimum. Therefore, the liquid surface tends to have the minimum area.

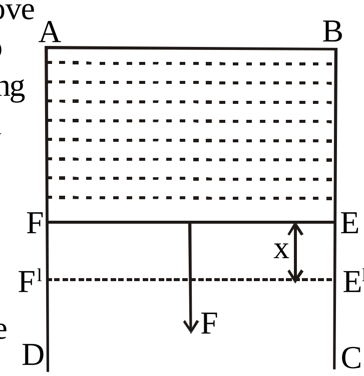
Note: Oil spreads on cold water. It may remain as drop on hot water, because, surface tension of oil is less than that of cold water but greater than that of hot water.

Surface energy

Surface energy may be defined as the PE per unit area of the liquid surface. OR it is the work done to increase the surface area by unity under isothermal conditions.

$$\text{Surface energy} = \frac{\text{Work done in increasing the surface area}}{\text{Increase in surface area}}$$

Relation between Surface Energy and Surface Tension



Consider a rectangular frame ABCD with a horizontal wire EF free to move up and down. Let a soap film ABEF be formed by dipping the frame in soap solution. The wire EF will be now pulled upward by the surface tension acting in the plane of the film and perpendicular to the wire. To keep the wire EF in position, a force equal to the upward force due to surface tension hs to be applied downwards. Let the downward force be equal to F. As the film has two surfaces the upward force due to surface tension,

$F = 2\ell S$; where ℓ = EF. $\{S = F/2\ell\}$

Now if the wire EF is pulled downwards through a small distance x to the position $E^{I}F^{I}$,

the work done = F x = $2\ell S x = S \times (2\ell x)$. Here $2\ell x$ is the increase in surface area of the soap film by considering both surfaces.

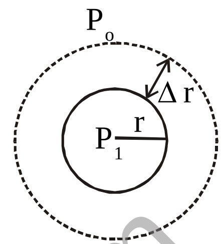
So work done = S x increase in surface area.

$$S = \frac{\text{work done}}{\text{increase in surface area}}$$

i.e. **Surface energy = surface tension**.

Drops and bubbles

Due to surface tension, a liquid surface always tend to have the minimum surface area. For a given volume, a sphere has the minimum surface area. Hence small drops and bubbles of a liquid assume spherical shape. However a big drop of a liquid is not spherical in shape, but is flattened. This is because, for small drops, the effective gravity is negligible and surface tension decides its shape. On the other hand, for bid drop, the effect of gravity predominates over surface area and the drop gets flattened.



Excess of pressure inside a spherical drop.

The molecules near the surface of a drop experience a resultant pull inwards due to surface tension. Due to this inward pull, the pressure inside is greater than that outside. Due to the force of surface tension, the drop tends to contract. But due to the excess of pressure inside over that outside, the drop tends to expand. When the drop is in equilibrium, these two forces will be equal and opposite.

Consider a drop of liquid of radius r. Let P_i and P_o be the values of pressure inside and outside the drop. Therefore, the excess of pressure inside the liquid drop over that outside is $P_i - P_o$.

Let the radius of the drop be increased by a small amount Δr under the pressure difference $P_i - P_o$. The outward force acting on the surface of liquid = excess of pressure x surface area.

i.e. $f = (P_i - P_o) 4\pi r^2$

The work done Δw when the radius increases by Δr is given by $\Delta w = F \Delta r = (P_i - P_o) 4\pi r^2 \Delta r$ ....(1)

The increase in surface area of the drop = $4\pi(r + \Delta r)^2 - 4\pi r^2 = 8\pi r^2 \Delta r$

{Since Δr is very small, we can neglect the term containing Δr^2}

If S is the surface tension of the liquid, the work done Δw to increase the surface area = S x increase in surface area.

$\Delta w = S \times 8\pi r \Delta r$ ....(2)

Equating (1) and (2) $(P_i - P_o) 4\pi r^2 \Delta r = S \times 8\pi r \Delta r$

$$\therefore \; P_i - P_o = \frac{2S}{r} \quad ....(3)$$

Excess of pressure inside a spherical bubble

A liquid bubble has air both inside and outside. Hence it has two surfaces - inner and outer. When the radius of the bubble increases, the area of both the surfaces will increase by $8\pi r \Delta r$.

Hence total increase in surface area = $16\pi r \Delta r$.

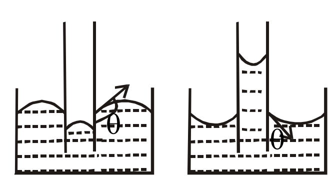
So excess of pressure, $P_i - P_o = \frac{4S}{r}$ in the case of liquid bubbles.

Excess of pressure inside an air bubble.

In the case of an air bubble, liquid is outside the bubble and air is inside it. It has only one free surface. So excess of pressure = $\frac{2S}{r}$.

Angle of contact

When a liquid meets a solid, the surface of the liquid in contact with solid is in general curved. *The angle between the tangent to the liquid surface, at the point of contact, and the solid surface inside the liquid is called the angel of contact for that point of solid and liquid*.



obtuse θ for Hg   acute θ for water

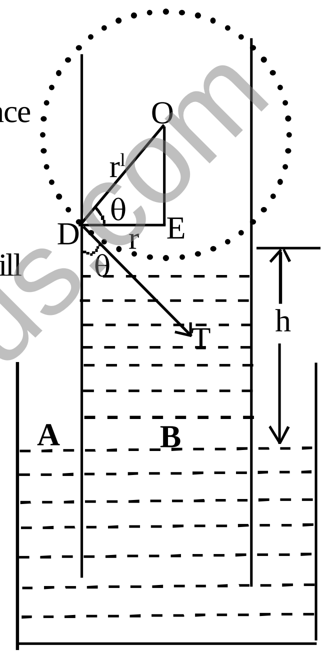
Angle of contact θ depends on the nature of the liquid and the solid. For liquids like water, which wet glass, the angle of contact is < 90^0

(acute), while for liquids like mercury which do not wet glass, it is > 90^0 (obtuse).

Capillary rise

A tube of very fine bore is called capillary tube. (meaning hair like tube). *When a clean capillary tube is dipped in a liquid, which wets it, the liquid immediately rises in the tube. This is called capillary rise*. The meniscus is found to be concave upwards for ordinary liquids. In the case of mercury, which does not wet glass, instead of capillary rise, a capillary depression is observed. In this case, the liquid meniscus is convex upwards.

Expression of capillary rise



Consider a capillary tube of radius r dipped vertically in a liquid of density ρ and surface tension S. The meniscus of the liquid inside the tube is concave. Hence the liquid rise through the tube up to a certain height. Let r^l be the radius of the meniscus, θ angle of contact and h the height of the liquid column in the tube with respect to the level of liquid outside.

The pressure just below the concave side of the liquid meniscus in the tube will be less than that above it by an amount $P = \frac{2S}{r^l}$. Pressure just below the meniscus in the tube $= P_o$., the atmospheric pressure.

So pressure at C, just below the meniscus = $P_0 - \frac{2S}{r^l}$

Since the pressure at the same horizontal level is same,

Pressure at A = Pressure at B

$$P_0 = P_0 - \frac{2S}{r^l} + h\rho g$$

$$h = \frac{2S}{r^l \rho g} = \frac{2S\cos\theta}{r\rho g}$$

In the right triangle ODE, OD = r^l, DE = r $\angle ODE = \theta$,

$$\cos\theta = \frac{DE}{DO} = \frac{r}{r^l} \qquad \therefore r^l = \frac{r}{\cos\theta}$$

In case of water and water like liquids which wet the tube, $\theta \approx 0$ as $\cos\theta = 1$ $\therefore h = \frac{2S}{r\rho g}$

Detergents and Surface tension

Dirty cloths containing grease and oil stains sticking on then cannot be cleaned by soaking them in water. This is because, water will not wet greasy dirt. But by adding detergent or soap to water, the greasy dirt can be removed. The molecules of the detergent are hair pin shaped. When the detergent is added to water, one end of the hair pin shaped molecules get attached to the water surface. When cloths having grease and oil stains are soaked in water containing detergents, the other end of the detergent molecules get attached to the molecules of the grease or oil. This result in formation of water grease interface, thereby reducing the surface tension between water and grease drastically. Now if the clothes are rinsed in water, the greasy dirt gets washed away by the moving water.

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