

MOTION OF SYSTEM OF PARTICLES AND RIGID BODY

K.V. Lumding; K.V. Karimganj; K.V. Langjing

CONCEPTS.

.Centre of mass of a body is a point where the entire mass of the body can be supposed to be concentrated

For a system of n -particles, the centre of mass is given by

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \dots + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \dots + m_n} = \frac{\sum_{i=1}^{i=n} m_i\vec{r}_i}{M}$$

.Torque τ The turning effect of a force with respect to some axis, is called moment of force or torque due to the force. Torque is measured as the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the axis of rotation.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

.Angular momentum (\vec{L}). It is the rotational analogue of linear momentum and is measured as the product of linear momentum and the perpendicular distance of its line of axis of rotation.

Mathematically: If \vec{P} is linear momentum of the particle and \vec{r} its position vector, then angular momentum of the particle, $\vec{L} = \vec{r} \times \vec{P}$

(a) In Cartesian coordinates : $L_z = xp_y - yp_x$

(b) In polar coordinates : $L = r p \sin\phi$,

Where ϕ is angle between the linear momentum vector \vec{P} and the position of vector \vec{r} .

S.I unit of angular momentum is $\text{kg m}^2\text{s}^{-1}$.

Geometrically, angular momentum of a particle is equal to twice the product of mass of the particle and areal velocity of its radius vector about the given axis.

.Relation between torque and angular momentum:

(i) $\vec{\tau} = \frac{d\vec{L}}{dt}$ (ii) If the system consists of n-particles, then $\vec{\tau} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \frac{d\vec{L}_3}{dt} + \dots + \frac{d\vec{L}_n}{dt}$.

.Law of conservation of angular momentum. If no external torque acts on a system, then the total angular momentum of the system always remain conserved.

Mathematically: $\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \dots + \vec{L}_n = \vec{L}_{total} = a \text{ constant}$

.Moment of inertia(I).the moment of inertia of a rigid body about a given axis of rotation is the sum of the products of masses of the various particles and squares of their respective perpendicular distances from the axis of rotation.

Mathematically: $I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2 = \sum_{i=1}^{i=n} m_i r_i^2$

SI unit of moment of inertia is kg m^2 .

MI corresponding to mass of the body. However, it depends on shape & size of the body and also on position and configuration of the axis of rotation.

Radius of gyration (K).it is defined as the distance of a point from the axis of rotation at which, if whole mass of the body were concentrated, the moment of inertia of the body would be same as with the actual distribution of mass of the body.

Mathematically : $K = \frac{r_1^2 + r_2^2 + r_3^2 + \dots + r_n^2}{n} = \text{rms distance of particles from the axis of rotation.}$

SI unit of gyration is m. Note that the moment of inertia of a body about a given axis is equal to the product of mass of the body and squares of its radius of gyration about that axis i.e. $I = Mk^2$.

.Theorem of perpendicular axes. It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moment of

inertia of the lamina about any two mutually perpendicular axes in its plane and intersecting each other at the point, where the perpendicular axis passes through the lamina.

Mathematically: $I_z = I_x + I_y$

Where x & y-axes lie in the plane of the Lamina and z-axis is perpendicular to its plane and passes through the point of intersecting of x and y axes.

.Theorem of parallel axes. It states that the moment of inertia of a rigid body about any axis is equal to moment of inertia of the body about a parallel axis through its center of mass plus the product of mass of the body and the square of the perpendicular distance between the axes.

Mathematically: $I = I_c + M h^2$, where I_c is moment of inertia of the body about an axis through its centre of mass and h is the perpendicular distance between the two axes.

.Moment of inertia of a few bodies of regular shapes:

- i. M.I. of a rod about an axis through its c.m. and perpendicular to rod,

$$I = \frac{1}{12} ML^2$$
- ii. M.I. of a circular ring about an axis through its centre and perpendicular to its plane, $I = MR^2$
- iii. M.I. of a circular disc about an axis through its centre and perpendicular to its plane, $I = \frac{1}{2} MR^2$
- iv. M.I. of a right circular solid cylinder about its symmetry axis, $I = \frac{1}{2} MR^2$
- v. M.I. of a right circular hollow cylinder about its axis = MR^2
- vi. M.I. of a solid sphere about its diameter, $I = \frac{2}{5} MR^2$
- vii. M.I. of spherical shell about its diameter, $I = \frac{2}{3} MR^2$

.Moment of inertia and angular momentum. The moment of inertia of a rigid body about an axis is numerically equal to the angular momentum of the rigid body, when rotating with unit angular velocity about that axis.

$$\text{Mathematically: } K.E \text{ of rotation} = \frac{1}{2} I \omega^2$$

.Moment of inertia and kinetic energy of rotation. The moment of inertia of a rigid body about an axis of rotation is numerically equal to twice the kinetic energy of rotation of the body, when rotation with unit angular velocity about that axis.

$$\text{Mathematically: } K.E. \text{ of rotation} = \frac{1}{2} I \omega^2$$

.Moment of inertia and torque. The moment of inertia of a rigid body about an axis of rotation is numerically equal to the external torque required to produce a unit angular acceleration in the body BOUT THE GIVEN AXIS.

$$\text{MATHEMATICALLY: } \tau = I a$$

.Law of conservation of angular momentum. If no external torque acts on a system, the total angular momentum of the system remains unchanged.

Mathematically:

*$I\omega = \text{constant vector, i. e., in magnitude, } I_1\omega_1 = I_2\omega_2,$
 provides no external torque acts on the system.*

For translational equilibrium of a rigid body, $\vec{F} = \sum_i F_i = 0$

For rotational equilibrium of a rigid body, $\vec{\tau} = \sum_i \vec{\tau}_i = 0$

1. The following table gives a summary of the **analogy** between various quantities describing linear motion and rotational motion.

s.no.	<u>Linear motion</u>	s.no.	<u>Rotational motion</u>
1.	Distance/displacement (s)	1.	Angle or angular displacement (θ)
2.	Linear velocity, $v = \frac{ds}{dt}$	2.	Angular velocity, $\omega = \frac{d\theta}{dt}$
3.	Linear acceleration, $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$	3.	Angular acceleration $= \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
	Mass (m)		Moment of inertia (I)
4.	Linear momentum, $p = m v$	4.	Angular momentum, $L = I\omega$
5.	Force, $F = m a$	5.	Torque, $\tau = I a$
6.	Also, force $F = \frac{dp}{dt}$	6.	Also, torque, $\tau = \frac{dL}{dt}$
7.		7.	
8.	Translational KE, $K_T = \frac{1}{2} m v^2$	8.	Rotational KE, $K_R = \frac{1}{2} I \omega^2$
9.	Work done, $W = F s$	9.	Work done, $W = \tau \theta$
	Power, $P = F v$		Power, $P = \tau \omega$

10.

Linear momentum of a system is conserved when no external force acts on the system.

11.

Equation of translator motion

12.

- i. $v = u + at$
- ii. $s = ut + \frac{1}{2}at^2$
- iii. $v^2 - u^2 = 2as$, where the symbols have their usual meaning.

10.

Angular momentum of a system is conserved when no external torque acts on the system

11.

Equations of rotational motion

12.

- i. $\omega_2 = \omega_1 + at$
- ii. $\theta = \omega_1 t + \frac{1}{2}at^2$
- iii. $\omega_2^2 - \omega_1^2 = 2a\theta$, where the symbols have their usual meaning.

CENTRE
OF MASS

CHARACTERISTICS

POSITION VECTOR

$$R = \frac{m_1 r_1 + m_2 r_2 + \dots + m_n r_n}{m_1 + m_2 + \dots + m_n}$$

COORDINATES

$$x = \frac{1}{M} \sum_{p=1}^N m_p x_p$$

$$y = \frac{1}{M} \sum_{p=1}^N m_p y_p$$

$$z = \frac{1}{M} \sum_{p=1}^N m_p z_p$$

MOTION

(IN CASE OF AN
ISOLATED SYSTEM)
UNIFORM VELOCITY

ROTATIONAL MOTION OF A PARTICLE IN A PLANE

CAUSES

TORQUE

CONSEQUENCES

ANGULAR
MOMENTUM

MOTION OF A STONE TIED
TO A STRING WOUND
OVER A ROTATING
CYLINDER

MOTION OF A BODY
ROLLING DOWN AN
INCLINED PLANE
WITHOUT SLIPPING

1 Marks Questions

1. If one of the particles is heavier than the other, to which will their centre of mass shift?

Answer:- The centre of mass will shift closer to the heavier particle.

2. Can centre of mass of a body coincide with geometrical centre of the body?

Answer:- Yes, when the body has a uniform mass density.

3. Which physical quantity is represented by a product of the moment of inertia and the angular velocity?

Answer:- Product of I and ω represents angular momentum ($L = I\omega$).

4. What is the angle between \vec{A} and \vec{B} , if \vec{A} and \vec{B} denote the adjacent sides of a parallelogram drawn from a point and the area of parallelogram is $\frac{1}{2}AB$.

Answer:- Area of parallelogram = $|\vec{A} \times \vec{B}| = AB \sin \theta = \frac{1}{2}AB$. (Given)

$$\sin \theta = \frac{1}{2} = \sin 30^\circ \text{ or } \theta = 30^\circ$$

5. Which component of linear momentum does not contribute to angular momentum?

Answer:- The radial component of linear momentum makes no contribution to angular momentum.

6. A disc of metal is melted and recast in the form of solid sphere. What will happen to the moment of inertia about a vertical axis passing through the centre ?

Answer:- Moment of inertia will decrease, because $I_d = \frac{1}{2} m r^2$ and $I_s = \frac{2}{5} m r^2$, the radius of sphere formed on recasting the disc will also decrease.

7. What is rotational analogue of mass of body?

Answer:- Rotational analogue of mass of a body is moment of inertia of the body.

8. What are factors on which moment of inertia depend upon?

Answer:- Moment of inertia of a body depends on position and orientation of the axis of rotation. It also depends on shape, size of the body and also on the distribution of mass of the body about the given axis.

9. Is radius of gyration of a body constant quantity?

Answer:- No, radius of gyration of a body depends on axis of rotation and also on distribution of mass of the body about the axis.

10. Is the angular momentum of a system always conserved? If no, under what condition is it conserved?

Answer:- No, angular momentum of a system is not always conserved. It is conserved only when no external torque acts on the system.

2 Marks Questions

1. Why is the handle of a screw made wide?

Answer:- Turning moment of a force = force \times distance (r) from the axis of rotation. To produce a given turning moment, force required is smaller, when r is large. That's what happens when handle of a screw is made wide.

2. Can a body in translatory motion have angular momentum? Explain.

Answer:- Yes, a body in translatory motion shall have angular momentum, the fixed point about which angular momentum is taken lies on the line of motion of the body. This follows from $|L| = r p \sin \theta$.

$L=0$, only when $\theta = 0^\circ$ or $\theta = 180^\circ$.

3. A person is sitting in the compartment of a train moving with uniform velocity on a smooth track. How will the velocity of centre of mass of compartment change if the person begins to run in the compartment?

Answer:- We know that velocity of centre of mass of a system changes only when an external force acts on it. The person and the compartment form one system on which no external force is applied when the person begins to run. Therefore, there will be no change in velocity of centre of mass of the compartment.

4. A particle performs uniform circular motion with an angular momentum L. If the frequency of particle's motion is doubled and its K.E is halved, what happens to the angular momentum?

Answer:- $L = m v r$ and $v = r \omega = r(2 \pi n)$

$$r = \frac{v}{2 \pi n} \quad \therefore \quad L = m v \left(\frac{v}{2 \pi n} \right) = \frac{m v^2}{2 \pi n}$$

As,

$$K.E = \frac{1}{2} m v^2, \text{ therefore, } L = \frac{K.E}{\pi n}$$

When K.E. is halved and frequency (n) is doubled, $L = \frac{K.E'}{\pi n'} = \frac{K.E/2}{\pi(2n)} = \frac{K.E}{4\pi n} = \frac{L}{4}$

i.e. angular momentum becomes one fourth.

5. An isolated particle of mass m is moving in a horizontal plane(x-y), along the x-axis at a certain height above the ground. It explodes suddenly into two fragments of masses m/4 and 3 m/4. An instant later, the smaller fragments is at y= +15 cm. What is the position of larger fragment at this instant?

Answer:- As isolated particle is moving along x-axis at a certain height above the ground, there is no motion along y-axis. Further, the explosion is under internal forces only. Therefore, centre of mass remains stationary along y-axis after collision. Let the co-ordinates of centre of mass be $(x_{cm}, 0)$.

$$\text{Now, } y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = 0 \quad \therefore \quad m_1 y_1 + m_2 y_2 = 0$$

$$\text{Or } y_2 = \frac{-m_1 y_1}{m_2} = \frac{-m/4}{3m/4} \times 15 = -5 \text{ cm}$$

So, larger fragment will be at $y = -5$; along x-axis.

6. Why there are two propellers in a helicopter?

Answer:- If there were only one propeller in a helicopter then, due to conservation of angular momentum, the helicopter itself would have turned in the opposite direction.

7. A solid wooden sphere rolls down two different inclined planes of the same height but of different inclinations. (a) Will it reach the bottom with same

speed in each case ? (b) Will it take longer to roll down one inclined plane than other ? Explain.

Answer:- (a) Yes, because at the bottom depends only on height and not on slope.

(b) Yes, greater the inclination(θ), smaller will be time of decent, as $t \propto 1/\sin \theta$.

8. There is a stick half of which is wooden and half is of steel. It is pivoted at the wooden end and a force is applied at the steel end at right angles to its length. Next, it is pivoted at the steel end and the same force is applied at the wooden end. In which case is angular acceleration more and why?

Answer:- We know that torque, $\tau = \text{Force} \times \text{Distance} = I \alpha = \text{constant}$

$$\therefore \alpha = \frac{\tau}{I} \quad \text{i.e. } \alpha \propto \frac{1}{I}$$

Angular acc. (α) will be more, when I is small, for which lighter material(wood) should at larger distance from the axis of rotation i.e. when stick is pivoted at the steel end.

9. Using expressions for power in rotational motion, derive the relation $P = I \alpha$, where letters have their usual meaning.

Answer:- We know that power in rotational motion, $P = \tau \omega$ (i)

$$\text{and K.E. of motion, } E = \frac{1}{2} I \omega^2$$

.....(ii)

As power = time rate of doing work in rotational motion, and work is stored in the body in the form of K.E.

$$\begin{aligned} \therefore P &= \frac{d}{dt} (\text{K.E. of rotation}) \\ &= \frac{d}{dt} \left(\frac{1}{2} I \omega^2 \right) = \frac{1}{2} I \times 2\omega \left(\frac{d\omega}{dt} \right) \\ P &= I \omega \alpha \end{aligned}$$

Using (i), $P = \tau \omega = I \omega \alpha$ or $\tau = I \alpha$, which is the required relation.

10. Calculate radius of gyration of a cylindrical rod of mass m and length L about an axis of rotation perpendicular to its length and passing through the centre.

Answer:- $K=?$, mass = m , length = L

Moment of inertia of the rod about an axis perpendicular to its length and passing through the centre is

$$I = \frac{mL^2}{12}$$

Also, $I = mK^2 \quad \therefore mK^2 = \frac{mL^2}{12} \quad \text{or} \quad K = \frac{L}{\sqrt{12}} = \frac{L}{2\sqrt{3}} .$

3 Marks Questions

1. Explain that torque is only due to transverse component of force. Radial component has nothing to do with torque.
2. Show that centre of mass of an isolated system moves with a uniform velocity along a straight line path.
3. If angular momentum is conserved in a system whose moment of inertia is decreased, will its rotational kinetic energy be also conserved ? Explain.

Ans:- Here, $L = I \omega = \text{constant}$

K.E. of rotation, $K = \frac{1}{2} I \omega^2$

$$K = \frac{1}{21} I^2 \omega^2 = \frac{L^2}{21}$$

As L is constant, $\therefore K \propto 1/I$

When moment of inertia(I) decreases, K.E. of rotation(K) increases. Thus K.E. of rotation is not conserved.

4. How will you distinguish between a hard boiled egg and a raw egg by spinning each on a table top?

Ans:- To distinguish between a hard boiled egg and a raw egg, we spin each on a table top. The egg which spins at a slower rate shall be raw. This is because in a raw egg, liquid matter inside tries to get away from its axis of rotation. Therefore, its moment of inertia I increases. As $\tau = I \alpha = \text{constant}$, therefore, α decreases i.e. raw egg will spin with smaller angular acceleration. The reverse is true for a hard boiled egg which will rotate more or less like a rigid body.

5. Equal torques are applied on a cylindrical and a hollow sphere. Both have same mass and radius. The cylinder rotates about its axis and the sphere rotates about one of its diameters. Which will acquire greater speed? Explain.

6. Locate the centre of mass of uniform triangular lamina and a uniform cone.

7. A thin wheel can stay upright on its rim for a considerable length when rolled with a considerable velocity, while it falls from its upright position at the slightest disturbance when stationary. Give reason.

Answer:- When the wheel is rolling upright, it has angular momentum in the horizontal direction i.e., along the axis of the wheel. Because the angular momentum is to remain conserved, the wheel does not fall from its upright position because that would change the direction of angular momentum. The wheel falls only when it loses its angular velocity due to friction.

8. Why is the speed of whirl wind in a tornado so high?

Answer:- In a whirl wind, the air from nearby region gets concentrated in a small space thereby decreasing the value of moment of inertia considerably. Since, $I\omega = \text{constant}$, due to decrease in moment of inertia, the angular speed becomes quite high.

9. Explain the physical significance of moment of inertia and radius of gyration.

10. Obtain expression for K.E. of rolling motion.

5 Marks Questions

1. Define centre of mass. Obtain an expression for perpendicular of centre of mass of two particle system and generalise it for particle system.

2. Find expression for linear acceleration of a cylinder rolling down on a inclined plane.

A ring, a disc and a sphere all of them have same radius and same mass roll down

on inclined plane from the same heights. Which of these reaches the bottom (i) earliest (ii) latest ?

3. (i) Name the physical quantity corresponding to inertia in rotational motion. How is it calculated? Give its units.

(ii) Find expression for kinetic energy of a body.

4. State and prove the law of conservation of angular momentum. Give one illustration to explain it.

5. State parallel and perpendicular axis theorem.

Define an expression for moment of inertia of a disc R, mass M about an axis along its diameter.

TYPICAL PROBLEMS

1. A uniform disc of radius R is put over another uniform disc of radius 2R of the same thickness and density. The peripheries of the two discs touch each other. Locate the centre of mass of the system.

Ans:-

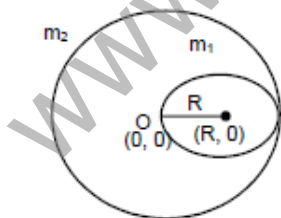
Let the centre of the bigger disc be the origin.

$2R =$ Radius of bigger disc

$R =$ Radius of smaller disc

$$m_1 = \pi R^2 \times T \times \rho$$

$$m_2 = \pi(2R)^2 \times T \times \rho, \text{ where } T = \text{Thickness of the two discs}$$



$\rho =$ Density of the two discs

\therefore The position of the centre of mass

$$= \left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \right)$$

$$\begin{array}{l} x_1 = R \\ x_2 = 0 \end{array} \quad \begin{array}{l} y_1 = 0 \\ y_2 = 0 \end{array}$$

$$\left(\frac{\pi R^2 T \rho R + 0}{\pi R^2 T \rho + \pi (2R)^2 T \rho}, \frac{0}{m^1 + m^2} \right) = \left(\frac{\pi R^2 T \rho R}{5 \pi R^2 T \rho}, 0 \right) = \left(\frac{R}{5}, 0 \right)$$

At $R/5$ from the centre of bigger disc towards the centre of smaller disc.

2. Two blocks of masses 10 kg and 20 kg are placed on the x-axis. The first mass is moved on the axis by a distance of 2 cm. By what distance should the second mass be moved to keep the position of centre of mass unchanged ?

Ans:- Two masses m_1 and m_2 are placed on the X-axis

$$m_1 = 10 \text{ kg} \quad , \quad m_2 = 20 \text{ kg}$$

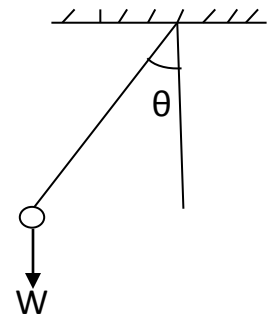
The first mass is displaced by a distance of 2 cm

$$\begin{aligned} \therefore \bar{X}_{cm} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{10 \times 2 + 20 x_2}{30} \\ &\Rightarrow 0 = \frac{20 + 20 x_2}{30} \\ &\Rightarrow 20 + 20 x_2 = 0 \\ &\Rightarrow 20 = -20 x_2 \\ &\Rightarrow x_2 = -1 \text{ cm} \end{aligned}$$

\therefore The 2nd mass should be displaced by a distance 1cm towards left so as to kept the position of centre of mass unchanged.

3. A simple of length l is pulled aside to make an angle θ with the vertical.

Find the magnitude of the torque of the weight w of the bob about the point of suspension. When is the torque zero ?



Ans:- A simple pendulum of length l is suspended from a rigid support.

A bob of weight W is hanging on the other point.

When the bob is at an angle θ with the vertical,

then total torque acting on the point of suspension = $\tau = F \times r$

$$\Rightarrow W r \sin \theta = W l \sin \theta$$

At the lowest point of suspension the torque will be zero as the force acting on the body passes through the point of suspension.

4. A square plate of mass 120 g and edge 5.0 cm rotates about one of edges. If it has a uniform angular acceleration of 0.2 rad/s^2 , what torque acts on the plate ?

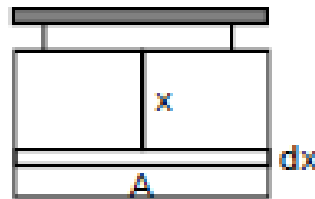
Ans:- A square plate of mass 120 gm and edge 5 cm rotates about one of the edge.

Let take a small area of the square of width dx and length a which is at a distance x from the axis of

rotation.

Therefore mass of that small area

$$m/a^2 \times a \, dx \quad (m = \text{mass of the square ; } a = \text{side of the plate})$$



$$I = \int_0^a (m/a^2) \times a x^2 dx = (m/a)(x^3/3) \Big|_0^a$$

$$= ma^2/3$$

$$\begin{aligned} \text{Therefore torque produced} &= I \times \alpha = (ma^2/3) \times \alpha \\ &= \{(120 \times 10^{-3} \times 5^2 \times 10^{-4})/3\} \times 0.2 \\ &= 0.2 \times 10^{-4} = 2 \times 10^{-5} \text{ N-m.} \end{aligned}$$

5. A wheel of moment of inertia 0.10 kg-m^2 is rotating about a shaft at an angular speed of 160 rev/minute. A second wheel is set into rotation at 300 rev/minute and is coupled to the same shaft so that both the wheels finally rotate with a common angular speed of 200 rev/minute. Find the moment of

inertia of the second wheel.

Ans:- Wheel (1) has

$$I_1 = 0.10 \text{ kg-m}^2, \omega_1 = 160 \text{ rev/min}$$

Wheel (2) has

$$I_2 = ? ; \omega_2 = 300 \text{ rev/min}$$

Given that after they are coupled, $\omega = 200 \text{ rev/min}$

Therefore if we take the two wheels to be an isolated system

Total external torque = 0

$$\text{Therefore, } I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$$

$$\Rightarrow 0.10 \times 160 + I_2 \times 300 = (0.10 + I_2) \times 200$$

$$\Rightarrow 5I_2 = 1 - 0.8$$

$$\Rightarrow I_2 = 0.04 \text{ kg-m}^2.$$

